The Lower and Upper Forcing Geodetic Numbers of Block-Cactus Graphs *

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Abstract

A vertex set D in graph G is called a geodetic set if all vertices of G are lying on some shortest $u-v$ path of G, where $u, v \in D$. The geodetic number of a graph G is the minimum cardinality among all geodetic sets. A subset S of a geodetic set D is called a forcing subset of D if D is the unique geodetic set containing S. The forcing geodetic number of D is the minimum cardinality of a forcing subset of D, and the lower and upper forcing geodetic numbers of a graph G are the minimum and maximum forcing geodetic numbers, respectively, among all geodetic sets of G. In this paper, we find out the lower and upper forcing geodetic numbers of block-cactus graphs.

Keyword: Geodetic set, Block-cactus graphs.

1 Introduction

All graphs considered in this paper are finite and simple (i.e., without loops and multiple edges). Let $G = (V, E)$ be a graph with vertex set V and edge set E. The cardinalities of V and E are called *order* and *size*, respectively. The *distance* of two vertices u and v in a connected graph G, denoted by $d(u, v)$, is the number of edges in a shortest $u-v$ path. A shortest $u-v$ path is also called a $u-v$ *geodesic*. Let $I(u, v)$ denote the set of all vertices lying on some $u-v$ geodesic of G, and $I(S) = \bigcup_{u,v \in S} I(u, v)$, where $S \subseteq V(G)$.

A vertex set D in graph G is called a *geodetic set* if $I(D) = V(G)$. A geodetic set with the minimum cardinality is said to be a *minimum geodetic set* (g-set for short). The cardinality of a g-set, denoted $g(G)$, is called the *geodetic number* of a graph G [12]. Notice that $2 \leq g(G) \leq |V|$ for $|V| \geq 2$. In [2], Buckley et al. characterized those connected graphs for which the geodetic number is equal to $|V|, |V| - 1$, or 2. Two classes of graphical games called *achievement* and *avoidance games* were examined for the geodetic number[1, 10]. Lately,

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Chartrand et al. boost the research on geodetic set problems [4, 5, 6, 7, 8] and determine the geodetic numbers for cycles, trees, etc. [8]. However, determining the geodetic number of a general graph is NP-hard [9].

A subset S of a g-set D is called a *forcing subset* of D if D is the unique g-set containing S. This means that D can be figured out after S is determined. A vertex in S is said to be a *forcing vertex* of D. The *forcing geodetic number* of D, denoted $f(D)$, is the minimum cardinality of a forcing subset for D [6]. The *upper forcing geodetic number*, denoted $f^+(G)$, of a graph G is the maximum forcing geodetic number among all g-sets of G [12]. Notice that $f^+(G) = 0$ if and only if G has exactly one g-set. In contrast, we define the *lower forcing geodetic number*, denoted $f^-(G)$, of a graph G to be the minimum forcing geodetic number among all g-sets of G. We use Figure 1 as an example to illustrate the above notation. In Figure 1, vertex set $\{a, b, c, d, e\}$ is intuitively a geodetic set of G. There are only three g-sets in G, namely $D_1 = \{a, b, e\}, D_2 = \{a, c, d\}$ and $D_3 = \{a, d, e\}$. Thus, $g(G) = 3$. Since D_1 is the only g-set containing b, it follows that $f(D_1) = 1$. Furthermore, D_2 is the only g-set containing c. Thus, $f(D_2) = 1$. In D_3, since every vertex of D_3 is also contained in some other g-set, $f(D_3) \geq 2$. It can be found that D_3 is the unique g-set containing $\{d, e\}$, and hence $f(D_3) = 2$. Therefore, $f^-(G) = \min\{f(D_1), f(D_2), f(D_3)\} = 1$ and $f^+(G) = \max\{f(D_1), f(D_2), f(D_3)\} = 2$. Moreover, the forcing number of a disconnected graph is defined to be the sum of the forcing numbers of all components. For simplicity, all the graphs considered in this paper are connected.

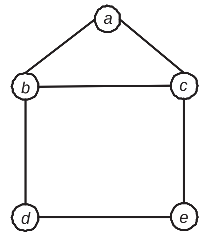

Figure 1: A graph G with $g(G) = 3$.

Researches on forcing concepts have been widely studied such as forcing domination number [3], forcing perfect matching [11] and forcing geodetic number [6]. Recently, Zhang determined the upper forcing geodetic numbers for trees, cycles, complete bipartite graphs and hypercubes [12]. In this paper, we furthermore find out the lower and upper forcing geodetic numbers of block-cactus graphs which are the general case of cycles and trees.

The remaining part of this paper is organized as follows. The next section introduces some basic terminologies, notation and previous results. In Section 3, we study the problem of finding the lower and upper forcing geodetic numbers on block-cactus graphs. Finally, we give concluding remarks and address our future researches in the last section.

2 Preliminaries and Previous Results

For any set S of vertices in a graph G, the subgraph *induced* by S, denoted by $[S]$, is the maximal subgraph of G with vertex set S. The induced subgraph $[V \setminus S]$ is denoted by $G - S$. It is the subgraph obtained from G by deleting the vertices in S together with their incident edges. If $S = \{v\}$, then we write $G - v$ for $G - \{v\}$. For a g-set D, the

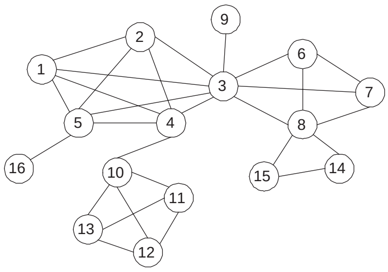

Figure 2: A block graph.

contribution of $[S]$ to $f(D)$ is the cardinality of a subset of S which is also a forcing subset of D. A vertex v is called an *extreme vertex* if the subgraph induced by the neighbors of v is complete. A vertex v is called a *cut vertex* if removing v and all edges incident to it increases the number of components. A *block* of a graph is a maximal subgraph without a cut vertex. A graph G is called a *block graph* if and only if every block of G is complete. Clearly, every vertex of a block graph is either a cut vertex or an extreme vertex. Figure 2 depicts a block graph in which vertices 3, 4, 5, 8 and 10 are cut vertices while other vertices are extreme vertices.

A block that is a cycle is called a *cyclic block*. A cyclic block B is *odd* (respectively, *even*) if the order of B is *odd* (respectively, *even*). A cyclic block with a unique cut vertex is called a *cyclic end-block* (*CEB* for short). We also call a block with more than one cut vertex a *cyclic internal-block* (*CIB* for short). A *cactus graph* is a graph in which every block with three or more vertices is a cyclic block. Furthermore, a graph whose blocks are either cycles or complete is called a *block-cactus graph*. Block-cactus graphs generalize the known classes of block graphs and cactus graphs [13]. For example, Figure 3 illustrates a cactus graph G with blocks $\{B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8\}$ and

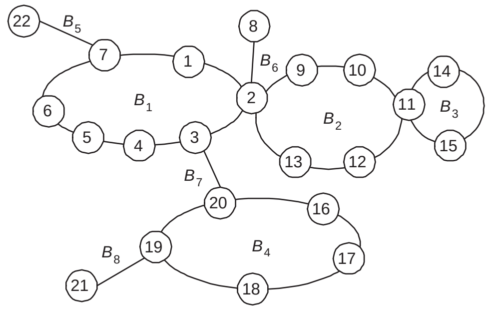

Figure 3: A cactus graph G.

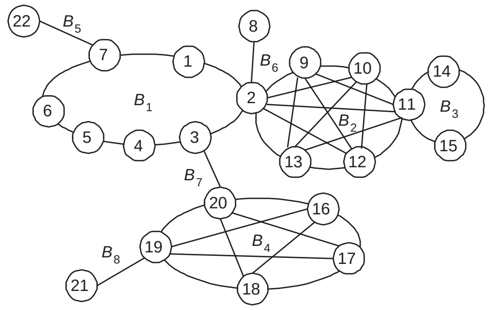

Figure 4: A block-cactus graph.

cut vertices are $2, 3, 7, 11, 19$ and 20. Blocks B_1, B_2 and B_4 are CIBs, while B_3 is a CEB. The orders of B_1, B_2, B_3 and B_4 are 7, 6, 3 and 5, respectively. Blocks B_1, B_3 and B_4 are odd and B_2 is even. Figure 4 is a block-cactus graph. We can see that B_2 and B_4 in Figure 3 are changed to complete graphs in Figure 4.

Consider CIBs of G and let B be a CIB of G. A path P in B is called a *segment* if both of its end vertices are cut vertices and the other vertices in P are not cut vertices. Clearly, $|P| < n$. A segment P is said to be a *long segment* if the length of P is greater than $\frac{n}{2}$. Intuitively, B has at most one long segment while other segments are called *short segments*. A CIB having a long segment is also called an *LCIB* except that the size of the CIB is odd and the length of the long segment is one less than the size of the CIB. We use $l(G)$ to denote the number of LCIBs in G. Take Figure 3 as an example. Block B_1 has three segments in which path: $3, 4, 5, 6, 7$ is a long segment while other segments are short. So, B_1 is an LCIB. Moreover, B_4 is another LCIB while B_2 is not. In B_4, segment $S_1 : 19, 20$ and $S_2 : 20, 16, 17, 18, 19$ are with the same end vertices. Nevertheless, S_1 is short and S_2 is long. Furthermore, since the length of S_2 is equal to the order of B_4 minus 1 and S_2 is in an odd CIB, S_2 is not an LCIB. Therefore, $l(G) = 1$.

The following lemmas shown by Zhang [12] are helpful to clarify our proof for determining the lower forcing geodetic numbers of other graphs.

Lemma 1 *For a graph G, $f^+(G) = 0$ if and only if G has exactly one g-set.*

For an integer $k \geq 2$, Zhang [12] showed that $g(C_{2k}) = 2$ and $g(C_{2k+1}) = 3$, where C_n is a cycle of n vertices. The next lemma describes the upper forcing geodetic numbers for C_n.

Lemma 2 $f^+(C_n) = \begin{cases} 0 & \text{if } n = 3, \\ 1 & \text{if } n \text{ is even}, \\ 2 & \text{if } n = 5, \\ 3 & \text{if } n \geq 7 \text{ is odd.} \end{cases}$

The next two lemmas state the inclusion and exclusion of extreme vertices and cut vertices, respectively, with respect to geodetic sets.

Lemma 3 *Every extreme vertex belongs to every geodetic set.*

Lemma 4 *If w is a cut vertex, then w cannot be a vertex of any g-set.*

By Lemma 3, every extreme vertex is certainly contained in every g-set. Therefore, the next corollary follows.

Corollary 5 *Every extreme vertex cannot be a forcing vertex.*

For a complete graph or a tree G, G has exactly one g-set which consists of all extreme vertices. Thus, by Corollary 5, we have the following corollary.

Corollary 6 *If G is a complete graph or a tree, then $f^-(G) = 0$.*

3 The Forcing Geodetic Numbers of Block-cactus Graphs

For the lower forcing geodetic number $f^-(G)$ of a graph G, if $f^-(G) = 0$, then G has exactly one g-set. Thus, the next lemma follows directly from Lemma 1.

Lemma 7 *The following statements are equivalent:*

(1) $f^+(G) = 0$,

(2) *G has exactly one g-set,*

(3) $f^-(G) = 0$.

For a block graph G, the geodetic number of G can be obtained directly from Lemmas 3 and 4. Furthermore, the lower and upper forcing geodetic numbers of G follow from Lemma 4 and Corollary 5.

Theorem 8 *Let c be the number of cut vertices of a block graph G. Then, $g(G) = |V| - c$ and $f^-(G) = f^+(G) = 0$.*

To determine the forcing geodetic numbers on cactus graphs, we need to find out the forcing geodetic numbers of cycles. In contrast with Lemma 2, we show the lower forcing geodetic number of a cycle as follows.

Lemma 9 $f^-(C_n) = \begin{cases} 0 & \text{if } n = 3, \\ 1 & \text{if } n \text{ is even}, \\ 2 & \text{if } n \text{ is odd and } n \neq 3. \end{cases}$

Proof. If $n = 3$, then every vertex is obviously an extreme vertex. By Lemma 3, $f^-(C_3) = 0$. For an even cycle $C_n : v_0, v_1, \ldots, v_{2k-1}, v_0$, every g-set is of the form $\{v_i, v_{(i+k) \bmod 2k}\}$ which is the only g-set containing v_i, where $0 \le i \le 2k-1$. Thus, $f^-(C_n) = 1$ for n is even.

At first, we show that $f^-(C_n) \ge 2$ for the case where n is odd and $n \ne 3$. It is clear that there are more than one g-set in C_n. By Lemma 7, $f^-(C_n) > 0$. We now show that $f^-(C_n) \ne 1$. Suppose to the contrary that there exists a g-set such that $f^-(C_n) = 1$. With vertex symmetric property of cycles, let $D = \{v_0, v_i, v_j\}, 0 \le i, j \le 2k-1$, be the unique g-set containing v_0. This implies that v_0 cannot be contained in any other g-set. Nevertheless, both $\{v_0, v_1, v_{k+1}\}$ and $\{v_0, v_k, v_{k+1}\}$ are also g-sets. It is a contradiction. Finally, we present a g-set D' with $f(D') = 2$ to complete the proof. Let $D' = \{v_0, v_1, v_{k+1}\}$ be a g-set of $C_n : v_0, v_1, \ldots, v_{2k}, v_0$. Clearly, D is the unique g-set containing $\{v_0, v_1\}$ and hence $f(D) = 2$. We conclude that $f^-(C_n) = 2$ if n is odd and $n \ne 3$.

Q. E. D.

Consider a CEB B of cactus graph G and let w be the cut vertex of B. If the order of B is 3, then all the vertices in $V(B) \setminus \{w\}$ are clearly extreme vertices, and hence are not forcing vertices. Therefore, We only need to consider the CEBs of order ≥ 4.

Lemma 10 *If B is a CEB of cactus graph G with order ≥ 4, then B contributes $f^-(B)-1$ to $f^-(G)$ and $f^+(B) - 1$ to $f^+(G)$.*

Proof. Let v be the cut vertex of B and D be a g-set of G. It can be seen that $G - B$ has at least one vertex, say u, in D. Let $D_v = D \cap V(B) \cup \{v\}$. Then, D_v is clearly a g-set of B. If the order of B is even, then, by Lemmas 2 and 9, B has exactly one forcing vertex in D_v. We can adjust this forcing vertex to be vertex v and this lemma follows. Now, we consider that the order of B is odd. Let S denote a forcing subset of D_v. Since the order of B is greater than 3, by Lemmas 2 and 9, $|S| \ge 2$. Without loss of generality, we can adjust S so that v is a vertex in S. Note that the vertices in $S \setminus \{v\}$ are still forcing vertices of D. This implies that B contributes $f^-(B) - 1$ to $f^-(G)$ and $f^+(B) - 1$ to $f^+(G)$.

Q. E. D.

The following notation will be used in Lemmas 11 and 12. Let P be a segment in CIB B of cactus graph G and the end vertices of P be a and b. Since a (respectively, b) is a cut vertex, $G - a$ (respectively, $G - b$) has at least two components. Let G_a (respectively, G_b) denote the subgraph which consists of all the components of $G - a$ (respectively, $G - b$) except the component containing P. It can be seen that G consists of G_a, G_b and B.

Lemma 11 *If w is a vertex of a short segment, then w cannot be a vertex of any g-set.*

Proof. Let P be a short segment in cycle B of cactus graph G, the end vertices of P be a and b and w be a vertex in P. It is clear that both G_a and G_b have at least one vertex in a g-set D of G. Let $a' \in G_a$ and $b' \in G_b$ be two vertices of D. Suppose to the contrary that w is in some g-set D. By definition, there is a vertex $u \notin D$ such that $u \in I(w, w')$ and $u \notin I(D \setminus \{w\})$ where $w' \in D$. Since P is short, $V(P) \subseteq I(a', b')$ and hence

$u \notin V(P)$. There are three cases to be considered depending on the position of u.

Case 1: $u \in G_a$.

Since $u \in I(w, w')$, w' must be in G_a. Therefore, u lies on a $a - w'$ geodesic P'. However, P' is indeed a subpath of a $b' - w'$ geodesic. Thus, u also lies on a $b' - w'$ geodesic. This contradicts that $u \notin I(D \setminus \{w\})$.

Case 2: $u \in G_b$.

This proof is similar to *Case* 1.

Case 3: $u \in B - P$.

Since $u \in I(w, w')$, w' must be in $B - P$. Let P' be a $w - w'$ geodesic where $u \in V(P')$. Clearly, $a - w$ or $b - w$ geodesic is a subpath of P'. If $a - w$ geodesic is a subpath of P', then $u \in I(a', w')$, a contradiction. Otherwise, $b - w$ geodesic must be a subpath of P'. Therefore, $u \in I(b', w')$. It is also a contradiction.

We conclude that w cannot be a vertex of any g-set.

Q. E. D.

Lemma 12 *Let B be a CIB with order n of cactus graph G and P be a long segment of B. Then, P has exactly one vertex in every g-set of G if and only if $|P| = n - 1$ and n is odd.*

Proof. Let $a' \in G_a$ and $b' \in G_b$ be two vertices of a g-set D of G. Since segment $P : a = v_1, v_2, \ldots, v_{|P|} = b$ is long, $V(P) \not\subseteq I(a', b')$. Thus, there is at least one vertex in $V(P) \cap D$ such that $V(P) \subseteq I(D)$. Moreover, since $V(P) \subseteq I(a', v_{\lfloor \frac{|P|}{2} \rfloor}) \cup I(b', v_{\lfloor \frac{|P|}{2} \rfloor})$, $|V(P) \cap D| = 1$. Let v_x be the vertex in $V(P) \cap D$. Then, x and $|P| - x$ are length of path:$v_1, v_2, \ldots, v_x$ and path:$v_x, v_{x+1}, v_{x+2}, \ldots, v_{n-1}$, respectively. Note that both x and $|P| - x$ must be less than or equal to $\lfloor \frac{n}{2} \rfloor$. At first, we prove that if $|P| = n - 1$ and n is odd, then v_x belongs to every g-set of G. Since n is odd, $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$. Then,

$$x \leq \frac{n-1}{2} \tag{1}$$

and

$$|P| - x \leq \frac{n-1}{2}. \tag{2}$$

According to Equations 1 and 2, we have $x = \frac{n-1}{2}$. Thus, v_x is the exactly one vertex that belongs to every g-set.

Now, we prove that if P has exactly one vertex v_x in every g-set of G, then $|P| = n - 1$ and n is odd. Since, by vertex symmetry property of cycle B, $v_{\frac{n-1}{2}}$ is the only possible vertex of B in D, $n-1$ is a multiple of 2. Therefore, n is odd. Suppose to the contrary that $|P| \leq n - 2$. Since $\frac{|P|}{2} + 1$ is less than $\frac{n}{2}$, $D \setminus \{v_{\frac{|P|}{2}}\} \cup \{v_{\frac{|P|}{2}+1}\}$ is also a g-set of G. This contradicts the fact that $v_{\frac{|P|}{2}}$ is the exactly one vertex in $V(P) \cap D$. We conclude that $|P| = n - 1$ and n is odd.

Q. E. D.

Consider a cactus graph G, if G is a cycle, then the forcing numbers of G are determined by Lemmas 2 and 9. For convenience, we use α and β to denote the cardinalities of the CEBs of order 5 and order ≥ 7, respectively.

Theorem 13 *If G is a cactus graph and not a cycle, then $f^-(G) = \alpha + \beta + l(G)$ and $f^+(G) = \alpha + 2 \cdot \beta + l(G)$.*

Proof. If $v \in V(G)$ does not belong to a CEB or a CIB of G, then v is clearly either an end vertex or a cut vertex. By Lemma 4 and Corollary 5, v cannot be a forcing vertex. Moreover, every even CEB contains no forcing vertex due to Lemmas 2, 9 and 10. Therefore, we next consider odd CEBs of order ≥ 5. By Lemmas 9 and 10, each odd CEB of order ≥ 5 contributes 1 to $f^-(G)$, while other CEBs have no contribution. Therefore, all CEBs totally contribute $\alpha + \beta$ to $f^-(G)$. Similarly, by Lemmas 2 and 10, each odd CEB of order 5 contributes 1 to $f^+(G)$ and each odd CEB of order ≥ 7 contributes 2 to $f^+(G)$. Thus, all CEBs totally contribute $\alpha + 2 \cdot \beta$ to $f^+(G)$.

Now we are at a position to consider the contribution of CIBs. Let P be a segment in CIB B of cactus graph G and the end vertices of P be a and b. Let D be a g-set of G. If P is short or $|P| = n-1$ and n is odd, then, by Lemmas 11 and 12, P has no contribution to $f(D)$. We then check LCIBs. By definition and Lemma 12, every LCIB contains exactly one vertex w in D, and w does not belong to all g-sets of G. That is, each LCIB contributes 1 to $f(D)$. Thus, all CIBs totally contribute $l(G)$ to $f(D)$.

Q. E. D.

For a block-cactus graph G, if $v \in V(G)$ does not belong to a cyclic block, then v must be an extreme vertex or a cut vertex. By Lemma 4 and Corollary 5, v cannot be a forcing vertex. Thus, we immediately sum up the contribution of all cyclic blocks for finding $f^-(G)$ and $f^+(G)$. Since the proof is similar to the proof of Theorem 13, we conclude the result as follows.

Theorem 14 *If G is a block-cactus graph and not a cycle, then $f^-(G) = \alpha + \beta + l(G)$ and $f^+(G) = \alpha + 2 \cdot \beta + l(G)$.*

4 Concluding Remarks

Zhang determined the upper forcing geodetic numbers for trees, cycles, complete bipartite graphs and hypercubes [12]. In contrast, we further propose another graph parameter, namely lower forcing geodetic number, and explore the lower and upper forcing geodetic numbers of block-cactus graphs. An obvious continuation of this work is to investigate the forcing geodetic numbers on other larger classes of graphs. Another line of progression will be to develop efficient algorithms for finding the set of forcing geodetic vertices.

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