DESIGN OF 2-D FAN FILTERS WITH 5X5 FILTERING MASKS

Tzon-Liang Shieh, Shih-Yu Chiu, and *Leu-Shing Lan*

Department of Electronics Engineering
National Yunlin University of Science and Technology, Taiwan

ABSTRACT

In digital image processing, filtering tasks are typically implemented using filtering masks of very small sizes, for instance, 3x3, 5x5, or 7x7. The choice of filtering coefficients are usually heuristic, without rigorous mathematical derivation. In this work, we propose a constrained least-squared (LS) error approach to design 5x5 image filtering masks, which have moderate degrees of freedom. The emphasis is on effective design of filtering masks of very small sizes. In specific, we concentrate on design of 2-D fan filters with quadrantal symmetry. Design formulas are explicitly derived. Some examples are provided to demonstrate the effectiveness of this principled design technique.

Keywords: image filtering masks, image filters, fan filters

1. INTRODUCTION

In digital image processing, very often we use a mask of very small size (for instance, 3x3, 5x5) to perform spatial filtering tasks. For instance, the averaging filter [1] and the binomial lowpass filter [2] are widely used in noise removal, which can be represented by their filtering masks as (for example, 3x3)

1/9	1/9	1/9		1/16	1/8	1/16
1/9	1/9	1/9		1/8	1/4	1/8
1/9	1/9	1/9		1/16	1/8	1/16

the averaging filter (3x3)   the binomial filter (3x3)

Nevertheless, conventional choice of the coefficients of filtering masks is based on *ad hoc* techniques. No rigorous optimization of these coefficients has been taken into account. From the viewpoint of digital filter design, these filters are not optimal filters in any sense.

In this paper, we present a constrained least-squared (LS) error approach to design compact-support image filtering masks. Specifically, we focus on the design of 2-D fan filters. Note that this problem is different from conventional digital filter design since the size of the filtering mask is required to be very small, where in this work we consider the 5x5 case. Due to speed and memory requirement, it is not wise to employ large-size filtering masks which incur expensive computational and memory costs. Fan filters are directional filters that are commonly used in 2-D multirate signal processing [5]. LS design approaches are, in general, very fast, with potential advantages to incorporate spatial-domain or frequency-domain constraints. For example, we can demand certain spatial symmetry requirements to reduce the design time. As another example, we may prespecify frequency responses at certain critical frequency points. Using the method of Lagrange multipliers, we can easily write down the design formulas in closed-form.

We also consider a special symmetry of filter coefficients in the spatial-domain, namely, the quadrantal symmetry [6, 7]. With quadrantal symmetry, filter coefficients of the 2nd, 3rd, and 4th quadrants are the same as those in the first quadrant, or more explicitly, the filtering masks are of the general form (for instance, 5x5)

$$
\begin{array}{ccccc}
i & h & g & h & i \\
f & e & d & e & f \\
c & b & a & b & c \\
f & e & d & e & f \\
i & h & g & h & i
\end{array}
$$

If this spatial symmetry is the only equality constraint, we can convert the original constrained LS design problem into an unconstrained one. Using this alternative formulation, we derived closed-form formulas for designing 2-D filters which exploit 5x5 filtering masks. Because of the quadrantal symmetry, the number of independent design parameters is reduced from 25 to 9.

This paper is organized as follows. In Section 2, the design problem is formulated as an equality-constrained optimization problem and then solved analytically in closed form. Section 3 presents an alternative solution as an unconstrained LS when the quadrantal symmetry property is utilized. Two specific design examples are given in Section 4. A conclusion section then follows.

2. PROBLEM DEFINITION AND SOLUTION

Although our final goal is to design small-size 2-D image filtering masks, we start from the 1-D formulation. With a

suitable arrangement of image data, the 2-D problem can be put into a 1-D form. Besides, 2-D separable filters can be obtained directly from 1-D filters (via a spatial convolution operation). Let the ideal frequency response be denoted by $d(k)$ and the realized frequency response be denoted by $\hat{d}(k)$, where $d(k)$ is the $(k+1)$st sample of the ideal continuous frequency response. Note that this sampling needs not to be uniform [3, 4]. We define the total squared error between $\hat{d}(k)$ and $d(k)$ by

$$\varepsilon \triangleq \sum_{k=0}^{N-1} \left| \hat{d}(k) - d(k) \right|^2 \tag{1}$$

where N is the number of samples of the ideal frequency response,

$$\hat{d}(k) = \sum_{n=-\frac{M-1}{2}}^{\frac{M-1}{2}} h(n)e^{-j\omega_k n} \tag{2}$$

$M+1$ is the number of filter coefficients, and ω_k is the $(k+1)$st sampling frequency. If we define

$$\mathbf{d} = [d(0), d(1), \ldots, d(N-1)]^T \tag{3}$$
$$\hat{\mathbf{d}} = [\hat{d}(0), \hat{d}(1), \ldots, \hat{d}(N-1)]^T \tag{4}$$
$$\mathbf{h} = [h(-\frac{M-1}{2}), h(-\frac{M-3}{2}), \ldots, h(\frac{M-1}{2})]^T \tag{5}$$

then the constrained optimization problem can be expressed as

$$\text{Minimize} \quad \varepsilon = (\mathbf{Wh} - \mathbf{d})^T(\mathbf{Wh} - \mathbf{d}) \tag{6}$$
$$\text{subject to} \quad \boldsymbol{Ah} = \boldsymbol{c} \tag{7}$$

where

$$W = \begin{bmatrix} e^{j\omega_0 \frac{M-1}{2}} & e^{j\omega_0 \frac{M-3}{2}} & \cdots & 1 \\ e^{j\omega_1 \frac{M-1}{2}} & e^{j\omega_1 \frac{M-3}{2}} & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ e^{j\omega_{N-1} \frac{M-1}{2}} & e^{j\omega_{N-1} \frac{M-3}{2}} & \cdots & 1 \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} \cdots & e^{-j\omega_0 \frac{M-1}{2}} \\ \cdots & e^{-j\omega_1 \frac{M-1}{2}} \\ \vdots & \vdots \\ \cdots & e^{-j\omega_{N-1} \frac{M-1}{2}} \end{bmatrix} \tag{9}$$

$\boldsymbol{A}$ is the constraint matrix, and $\boldsymbol{c}$ is the constraint vector. This equality-constrained minimization problem can be solved easily using the method of Lagrange multipliers. We summarize the results as

$$\mathbf{h} = (\mathbf{W}^T\mathbf{W})^{-1}(\mathbf{W}^T\mathbf{d} - \frac{1}{2}\mathbf{A}^T\lambda) \tag{10}$$
$$\lambda = 2[\mathbf{A}(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{A}^T]^{-1}[\mathbf{A}(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{W}^T\mathbf{d} - \mathbf{c}] \tag{11}$$

The dimension of the quantities are $\mathbf{h} : M \times 1$, $\mathbf{W} : N \times M$, $\mathbf{d} : N \times 1$, $\mathbf{A} : R \times M$, $\lambda : R \times 1$, and $\mathbf{c} : R \times 1$. Once the number of filter coefficients, M, is given, $\mathbf{A}$ and $\mathbf{W}$ can be determined immediately, and λ is then obtained using Eq. (10) according to the specified $\mathbf{d}$, the desired frequency response. The optimal filter coefficients $\mathbf{h}$ are then obtained by substituting λ into Eq. (11). To simplify our design, we consider a uniform sampling of the desired frequency response to obtain $\mathbf{d}$. However, the method proposed in this paper can also be applied for the nonuniform case.

In 2-D design, the same problem formulation can be used. However, we need to first convert the 2-D frequency response matrix into a long 1-D vector so that Eqs. (10) and (11) can be directly applied. Since this procedure is straightforward, the details are not shown here.

3. AN ALTERNATIVE UNCONSTRAINED FORMULATION

In this work we focus on 2-D 5x5 filtering masks with quadrantal symmetry, which are of form

$$\begin{matrix} h_{22} & h_{12} & h_{02} & h_{12} & h_{22} \\ h_{21} & h_{11} & h_{01} & h_{11} & h_{21} \\ h_{20} & h_{10} & h_{00} & h_{10} & h_{20} \\ h_{21} & h_{11} & h_{01} & h_{11} & h_{21} \\ h_{22} & h_{12} & h_{02} & h_{12} & h_{22} \end{matrix} \tag{12}$$

that has only nine independent design parameters. For such a quadrantally symmetric filtering mask, the frequency response is given by

$$\begin{aligned} H(e^{j\omega_1, j\omega_2}) = \quad & h_{00} + 2h_{10}\cos\omega_1 + 2h_{20}\cos(2\omega_1) \\ + \quad & 2h_{01}\cos\omega_2 + 2h_{02}\cos(2\omega_2) \\ + \quad & 2h_{11}(\cos(\omega_1 + \omega_2) + \cos(\omega_1 - \omega_2)) \\ + \quad & 2h_{21}(\cos(2\omega_1 + \omega_2) + \cos(2\omega_1 - \omega_2)) \\ + \quad & 2h_{12}(\cos(\omega_1 + 2\omega_2) + \cos(\omega_1 - 2\omega_2)) \\ + \quad & 2h_{22}(\cos(2\omega_1 + 2\omega_2) + \cos(2\omega_1 - 2\omega_2)) \end{aligned} \tag{13}$$

The implemented frequency response vector $\hat{\mathbf{d}}$ can thus be written as

$$\hat{\mathbf{d}} = \boldsymbol{W}\mathbf{h}_r \tag{14}$$

where the reduced coefficient vector $\mathbf{h}_r$ is given by

$$\mathbf{h}_r = [h_{00}, h_{10}, h_{20}, h_{01}, h_{02}, h_{11}, h_{21}, h_{12}, h_{22}]^T \tag{15}$$

and the ith-row of $\boldsymbol{W}$ is denoted as

$$\boldsymbol{W}_i \triangleq [1 \ b_2 \ b_3 \ b_4 \ b_5 \ b_6 \ b_7 \ b_8 \ b_9] \tag{16}$$

with

$$
\begin{aligned}
b_2 &= 2\cos\omega_1 \\
b_3 &= 2\cos(2\omega_1) \\
b_4 &= 2\cos\omega_2 \\
b_5 &= 2\cos(2\omega_2) \\
b_6 &= 2(\cos(\omega_1 + \omega_2) + \cos(\omega_1 - \omega_2)) \\
b_7 &= 2(\cos(2\omega_1 + \omega_2) + \cos(2\omega_1 - \omega_2)) \\
b_8 &= 2(\cos(\omega_1 + 2\omega_2) + \cos(\omega_1 - 2\omega_2)) \\
b_9 &= 2(\cos(2\omega_1 + 2\omega_2) + \cos(2\omega_1 - 2\omega_2)) \quad (17)
\end{aligned}
$$

where the frequency grid spans

$$
\omega_1 \in \left[-\pi, \frac{(N_1 - 1)\pi}{N_1}\right] \quad (18)
$$

$$
\omega_2 \in \left[-\pi, \frac{(N_2 - 1)\pi}{N_2}\right] \quad (19)
$$

with N_1 and N_2 being the number of frequency sampling points in the ω_1 and ω_2 axes, respectively.

Since we have converted the original equality-constrained optimization problem into an unconstrained minimization problem, the solutions are readily obtained as

$$
\mathbf{h}_r = (\boldsymbol{W}^T\boldsymbol{W})^{-1}\boldsymbol{W}^T\mathbf{d}. \quad (20)
$$

4. DESIGN EXAMPLES

This work focuses on the design of 2-D fan filters using small-size filtering masks with quadrantal symmetry. We give two examples to demonstrate the characteristics of the designed filters. In both examples, the number of frequency grid points used in the design is 100×100.

Example 1 The magnitude spectrum specifications of this example is shown in Fig. 1 with the resultant 5x5 filtering mask depicted in Table 1. The magnitude response and contour plot of the designed filter are shown in Figs. 2(a) and (b), respectively.

Example 2 The magnitude spectrum specifications of this example is shown in Fig. 3 with the resultant 5x5 filtering mask depicted in Table 2. The magnitude response and contour plot of the designed filter are shown in Figs. 4(a) and (b), respectively.

5. CONCLUSIONS

This article proposes a constrained least-squared error approach to design compact-support image filtering masks. This approach is different from more sophisticated digital filter design techniques since we require that the number of filter coefficients to be very limited. On the other hand,

the proposed approach also distinguishes it from conventional image filtering masks since the latter is usually chosen heuristically. We have demonstrated the effectiveness of the new method through designing two fan filters. Other type of filters can as well be designed without any difficulty as long as they satisfy the quadrantaly symmetry property.

6. REFERENCES

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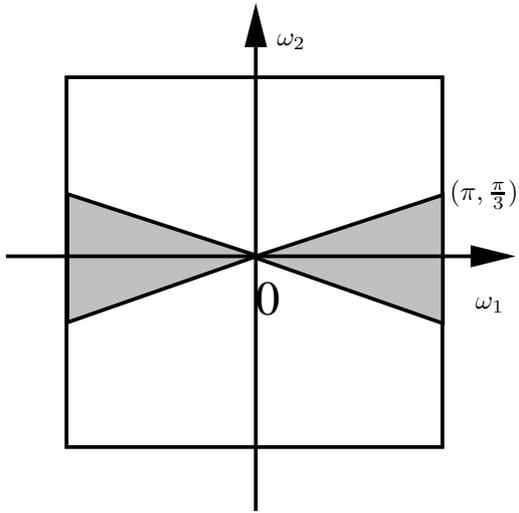
Fig. 1. Specifications of the fan filter in Example 1.

Table 1. **The filtering mask of Example1.**

-0.0143	-0.0298	0.1147	-0.0298	-0.0143
-0.0044	-0.0568	0.1534	-0.0568	-0.0044
0.0000	-0.0676	0.1684	-0.0676	0.0000
-0.0044	-0.0568	0.1534	-0.0568	-0.0044
-0.0143	-0.0298	0.1147	-0.0298	-0.0143

(a)

(b)
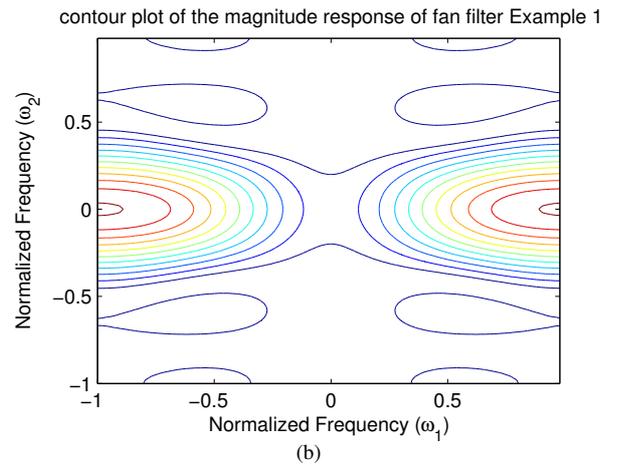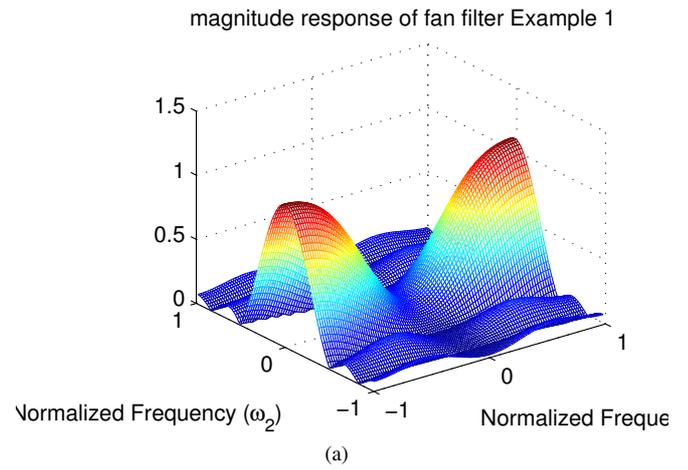
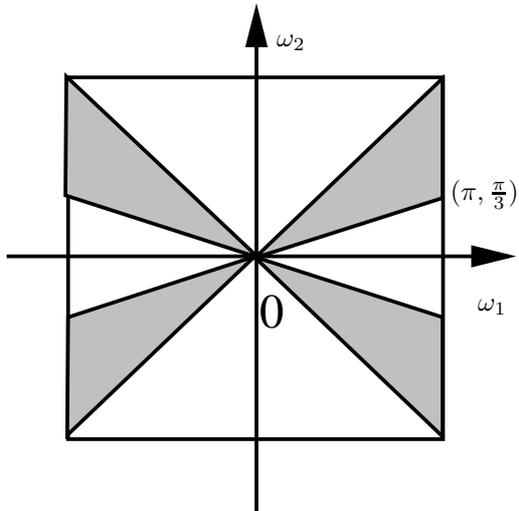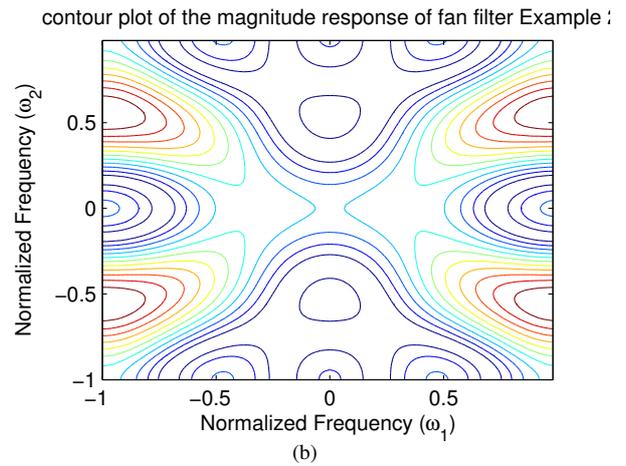**Fig. 2**. (a) The magnitude response of the designed filter in Example 1. (b) The corresponding contour plot.

Fig. 3. Specifications of the fan filter in Example 2.

Table 2. **The filtering mask of Example2.**

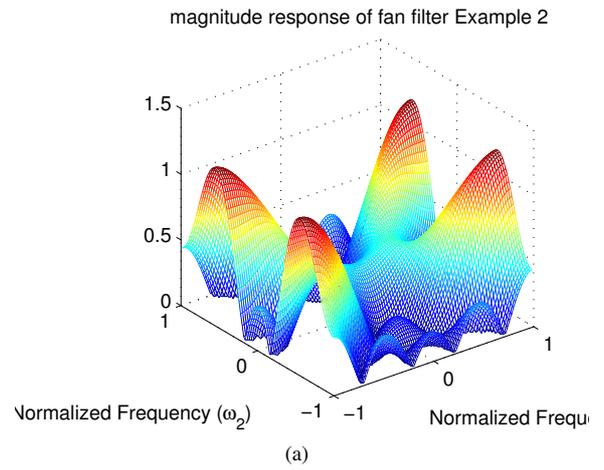|				
0.0166	0.0985	-0.1133	0.0985	0.0166
-0.0633	0.0596	0.0520	0.0596	-0.0633
-0.0000	-0.1351	0.3400	-0.1351	-0.0000
-0.0633	0.0596	0.0520	0.0596	-0.0633
0.0166	0.0985	-0.1133	0.0985	0.0166

(a)

(b)

Fig. 4. (a) The magnitude response of the designed filter in Example 2. (b) The corresponding contour plot.