Fault tolerance for Hamiltonian cycle of node expansion on hypercube

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Abstract

In this paper, we construct the variant of hypercube $X(Q_n, \{x_b, x_w\})$ *with node expansion on one black node* x_b *and one white node* x_w *of hypercube* $Q_n = (V_b \cup V_w, E)$. *Let* $F = F_b \cup F_w \cup F'$ *be the faulty set of* $X(Q_n, \{x_b, x_w\})$ *where* $F_b \subset V_b$, $F_w \subset V_w$ *and* F' *are disjoint sets. We show that* $X(Q_n, \{x_b, x_w\}) - F$ *is Hamiltonian if* (1).$|F_b| = |F_w| = 0, |F'| \leq n-2$, (2).$0 < |F_b| = |F_w| \leq \lceil \frac{n}{4} \rceil - 1, |F'| \leq n - 1 - 4|F_b|$, (3).$0 \leq |F_w| \neq |F_b| \leq \lceil \frac{n}{4} \rceil - 2, |F'| \leq n-3-4f_{max}$, *for* $f_{max} = \max\{|F_b|, |F_w|\}$. *We thus derive that* $X(Q_n, x_b, x_w)$ *is k-Hamiltonian for* $k = \lceil \frac{n}{4} \rceil - 2$. *We also investigate the fault tolerance for multi-spanning disjoint paths of complete graph* K_n *and hypercube* Q_n.

Keywords: Hypercube; Node expansion; Fault-tolerant; k-Hamiltonian; Spanning disjoint paths.

1 Introduction

The hypercube is a popular and efficient interconnection network. It has been widely use due to many excellent properties, such as regularity, symmetry, low diameter and degree, effective and simple routing, and so on. Component failures are unavoidable in a large parallel systems. Therefore, fault tolerance of an interconnection network is very important research issue.

The interconnection network can be expressed as a graph. The vertices on the graph represent processors and edges represent link between processors. Let $G = (V, E)$ be an undirected graph, where $V(G)$ is the node set and $E(G)$ is the edge set. The degree of a vertex v is the number of edges adjacent to v denote $d_G(v)$. A *Hamiltonian cycle*(resp. *Hamiltonian path*) is a cycle(resp. path) of a graph that visits every vertex exactly once. A graph G is a *Hamiltonian graph* if there is a Hamiltonian cycle of G. A graph $G = (V, E)$ is k-Hamiltonian if $G - F$ is Hamiltonian for $F \subset (V \cup E)$ and $|F| \leq k$. Some variants of hypercube are $(n-2)$-Hamiltonian graphs [2, 3, 6].

A graph $G = (V_b \cup V_w, E)$ is a *bipartite graph* if each edge of E consists of one vertex from the white vertex set V_w and one vertex from the black vertex set V_b. A bipartite graph $G = (V_b \cup V_w, E)$ is *Hamiltonian laceable* if there exists a Hamiltonian path between b, w for any $b \in V_b, w \in V_w$. In [7], Tsai et al. proved that the hypercube $Q_n - F_e$ is Hamiltonian laceable for $F_e \subset E(Q_n)$ and $|F_e| \leq n-2$.

In [5], the authors investigated the vertices fault-tolerance for multiple spanning disjoint paths for hypercube $Q_n = (V_b \cup V_w, E)$. Let $F_b \subset V_b$ and $F_w \subset V_w$ be two sets of faulty vertices of Q_n. Let $K_b \subset (V_b - F_b)$ and $K_w \subset (V_w - F_w)$ be two sets of fault-free vertices of Q_n for $|K_b| + |K_w|$ is even. Let $K_b \cup K_w = \{s_i, t_i|$ for $1 \leq i \leq \frac{|K_b|+|K_w|}{2}\}$. The family $\{s_i, t_i\}^{F_b,K_b}_{F_w,K_w}$ is *connectable* if there exist $\frac{|K_b|+|K_w|}{2}$ spanning disjoint paths $P(s_i, t_i)$, for $1 \leq i \leq \frac{|K_b|+|K_w|}{2}$, in $Q_n - F_b - F_w$. The family $\{s_i, t_i\}^{F_b,K_b}_{F_w,K_w}$ is *balanced* if $|K_w| + 2|F_w| = |K_b| + 2|F_b|$. Hung et al. proved that every balanced family $\{s_i, t_i\}^{F_b,K_b}_{F_w,K_w}$ of hypercube Q_n is connectable if $|F_b|+|F_w|+|K_b|+|K_w| \leq n$, $4|F_b|+2|K_b| = 4|F_w|+2|K_w| \leq n+1$, for $n \geq 3$.

Hung et al. presented the *t-node expansion* for k-Hamiltonian graph in [4]. Let X_n be the graph obtained by applying n-node expansion to every vertex of hypercube Q_n. The authors proved that X_n is $(n-2)$-Hamiltonian in [4].

In this paper, we will prove the vertices and edges fault-tolerance for multiple spanning disjoint paths of hypercube Q_n. Let F_e be the set of faulty edges of Q_n. We will show that every balanced family $\{s_i, t_i\}^{F_b,K_b}_{F_w,K_w}$ of hypercube $Q_n - F_e$ is connectable if $|F_b| + |F_w| + |K_b| + |K_w| + |F_e| \leq n$, $4|F_b| + 2|K_b| + |F_e| = 4|F_w| + 2|K_w| + |F_e| \leq n+1$, for $n \geq 3$.

Let $x_b \in V_b$ and $x_w \in V_w$ be any two vertices of Q_n. Let $X(Q_n, \{x_b, x_w\})$ be the graph obtained by n-node expansion on x_b and x_w of Q_n. Let $F =$

$F_b \cup F_w \cup F'$ be the faulty set of $X(Q_n, \{x_b, x_w\})$ where $F_b \subset V_b, F_w \subset V_w$ and F' are disjoint sets. We will prove that $X(Q_n, \{x_b, x_w\}) - F$ is Hamiltonian if

1. $|F_b| = |F_w| = 0, |F'| \leq n-2$,
2. $0 < |F_b| = |F_w| \leq \lceil \frac{n}{4} \rceil - 1, |F'| \leq n-1-4|F_b|$,
3. $0 \leq |F_w| < |F_b| \leq \lceil \frac{n}{4} \rceil - 2, |F'| \leq n-3-4|F_b|$.

Applying this result, we prove that $X(Q_n, \{x_b, x_w\})$ is k-Hamiltonian for $k = \lceil \frac{n}{4} \rceil - 2$.

The rest of this paper is organized as follows. In section 2, we show fault tolerance for spanning disjoint paths of complete graphs. We will prove the vertices and edges fault-tolerance for multiple spanning disjoint paths of hypercube in section 3. In section 4, the fault tolerance for Hamiltonian cycle of node expansion on hypercube is proved. The conclusion is given in section 5.

2 Fault tolerance for spanning disjoint paths of complete graph

Hung et al. proved the following lemma in [4]

Lemma 1 *Let $K_n = (V, E)$ be an n-node complete graph and $F \subset (V \cup E)$ be a faulty set with $|F| \leq n-2$. There exists a set $V' \subseteq V(K_n - F)$ with $|V'| = n - |F|$ such that every pair of vertices in V' can be joined by a Hamiltonian path.*

The following theorem is the generalization of Lemma 1.

Theorem 1 *Let $K_n = (V, E)$ be an n-node complete graph and $F \subset (V \cup E)$ be a faulty set with $|F| \leq n-2$. There exists a set $V' \subseteq V(K_n - F)$ with $|V'| = n - |F|$. Such that any m pairs of vertices in V', there exist m spanning disjoint paths of $K_n - F$ for $1 \leq m \leq \lfloor \frac{n-|F|}{2} \rfloor$.*

Proof:

We prove this theorem by induction on n. Trivially, this theorem is true for $|F| = 0$, Applying Lemma 1, we can obtain that this theorem holds for $m = 1$. In the following, we can assume that $|F| \geq 1$ and $m \geq 2$. Thus, $|V'| = n - |F| \geq 4$. Hence, we can assume $n \geq 5$.

First, we consider $|F \cap V(K_n)| > 0$. Let F_v denote the set of faulty nodes. Then, the graph $K_n - F$ is isomorphic to $K_{n-|F_v|} - F'$, $|F'| \leq |F| - |F_v|$. By induction hypotheses, there exists a set $V' \subseteq V(K_{n-|F_v|} - F')$ with $|V'| = n - |F_v| - |F'| \geq n - |F|$. Such that any m pairs of vertices in V', there exist m spanning disjoint paths of $K_n - F$ for $2 \leq m \leq \lfloor \frac{n-|F_v|-|F'|}{2} \rfloor$. This theorem is true for $|F \cap V(K_n)| > 0$.

Next, we consider that $F \subset E$. We only need to consider that $F \subset E$ and $|F| \leq n-4$. Let H denote the subgraph of K_n given by (V, F). Let $U = \{x | x \in V \text{ and } d_H(x) > 0\}$ and v be the vertex in U with minimum degree. We will prove this theorem with the following three cases:

Case 1: $d_H(v) = 1$.

In other words, there is exactly one edge of F incident to v. Thus, the graph $K_n - \{v\} - F$ is isomorphic to $K_{n-1} - F^*$ with $|F^*| \leq |F| - 1$. By induction hypotheses, there exists a vertex set $V' \subset (V - \{v\})$ with $|V'| = n - 1 - |F^*|$. Such that any m pairs of vertices in V', there exist m spanning disjoint paths of $K_n - \{v\} - F^*$ for $2 \leq m \leq \lfloor \frac{n-1-|F^*|}{2} \rfloor \leq \lfloor \frac{n-|F|}{2} \rfloor$. Since $d_H(v) = 1$ and $m \geq 2$, there exists an edge (z_1, z_2) of one of these path, such that $(v, z_1), (v, z_2) \notin F$. Hence, we can modify this path by replacing (z_1, z_2) by (z_1, v) and (v, z_2), as illustrated in Figure 1. Therefore, there exists a set $V' \subseteq V$ with $|V'| = n - |F|$. Such that any m pairs of vertices in V', there exist m spanning disjoint paths of $K_n - F$ for $1 \leq m \leq \lfloor \frac{n-|F|}{2} \rfloor$.

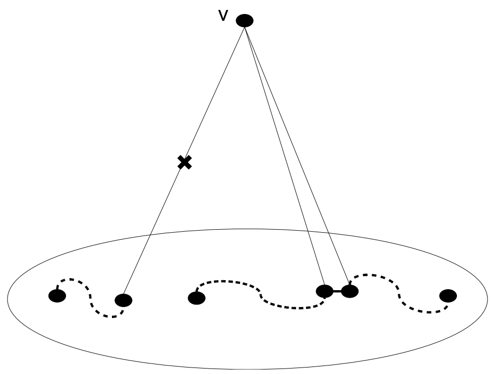

Figure 1: Illustration of **Case1**.

Case 2: $d_H(v) = 2$.

Since v is the vertex in U with minimum degree and $d_H(v) = 2$, $|F| \geq 3$. Thus, $n \geq |F| + 4 \geq 7$. The graph $K_n - \{v\} - F$ is isomorphic to $K_{n-1} - F^*$ with $|F^*| = |F| - 2$. By induction hypotheses, there exists a vertex set $V' \subset (V - \{v\})$ with $|V'| = n - |F| + 1$. For every m pairs of vertices in V', there exist m spanning disjoint paths of $K_n - \{v\} - F^*$. Let $x, y \in (V - \{v\})$ and $(v, x), (v, y) \in F$.

First, we consider that $n = 7$. Since $|F| \leq n - 4$, $|F| = 3$. Thus, $V' \cap \{x, y\} \neq \emptyset$. Without loss of generality, we can assume that $x \in (V' \cap \{x, y\})$. We will choose two pairs of vertices from V'. Let $V^* \subset V'$ and $x \in V^*$ with $|V^*| = 4$. There exist two spanning disjoint paths of $K_n - \{v\} - F^*$ between every pair of vertices in V^*. Hence there exists an edge (z_1, z_2) of these two paths such that $\{z_1, z_2\} \cap \{x, y\} = \emptyset$. Thus, we can modify the path by replacing (z_1, z_2) by (z_1, v) and (v, z_2). Therefore, there exists a vertex set V'

with $|V'| = n - |F|$. Such that every m pairs of vertices in V', there exist m spanning disjoint paths of $K_n - F$ for $1 \leq m \leq \lfloor \frac{n-|F|}{2} \rfloor$. This theorem is true for $n = 7$.

Next, we will consider that $n \geq 8$. Suppose that $m = 2$. The number of edges of these 2 spanning disjoint paths of $K_n - \{v\} - F^*$ is at least 5. Suppose that $m \geq 3$. Since $d_H(v) = 2$, there exists one of theses m spanning disjoint paths of $K_n - \{v\} - F^*$ such that every vertex of this path is adjacent to v. Thus, there exists an edge (z_1, z_2) of some path, such that $(v, z_1), (v, z_2) \notin F$. Hence, we can modify this path by replacing (z_1, z_2) by (z_1, v) and (v, z_2), as illustrated in Figure 2. Therefore, there exists a set $V' \subseteq V$ with $|V'| = n - |F|$. Such that any m pairs of vertices in V', there exist m spanning disjoint paths of $K_n - F$ for $1 \leq m \leq \lfloor \frac{n-|F|}{2} \rfloor$.

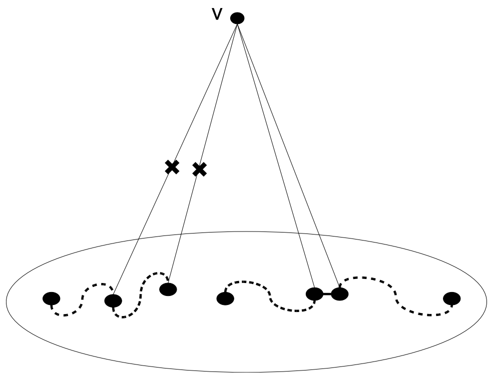

Figure 2: Illustration of **Case2**.

Case 3: $d_H(v) \geq 3$.
The graph $K_n - \{v\} - F$ is isomorphic to $K_{n-1} - F^*$ with $|F^*| = |F| - 3$. By induction hypothesis, there exists a set $V' \subset (V - \{v\})$ with $|V'| = n - |F| + 2$. For every m pairs of vertices of V', there exist m spanning disjoint paths of $K_n - \{v\} - F^*$ for $1 \leq m \leq \lfloor \frac{n-|F|+2}{2} \rfloor$. Since v is the vertex of U with minimum degree, $|F| \geq \frac{d_H(v) \cdot (d_H(v)+1)}{2}$. The number of edges of m spanning disjoint paths in $K_n - \{v\} - F^*$ is $n - m - 1$. Thus, $n - m - 1 \geq |F| + 2m - m - 1 \geq \frac{d_H(v) \cdot (d_H(v)+1)}{2} + m - 1 > 2d_H(v)$ for $d_H(v) \geq 3$ and $m \geq 2$. Thus, there exists an edge (z_1, z_2) of one of these spanning disjoint paths such that $(v, z_1), (v, z_2) \notin F$. Hence, we can modify this path by replacing (z_1, z_2) by (z_1, v) and (v, z_2), as illustrated in Figure 3. Therefore, there exists a set $V' \subseteq V$ with $|V'| = n - |F|$. Such that any m pairs of vertices in V', there exist m spanning disjoint paths of $K_n - F$ for $1 \leq m \leq \lfloor \frac{n-|F|}{2} \rfloor$. □

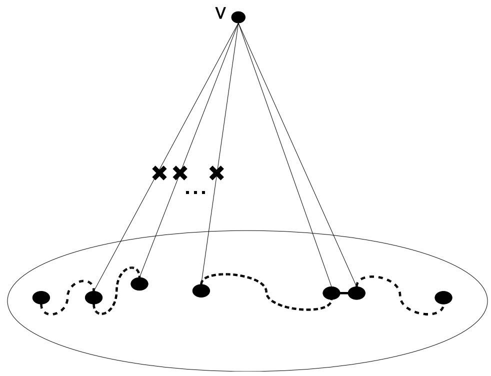

Figure 3: Illustration of **Case3**.

3 Fault tolerance for spanning disjoint paths of hypercube

In this section, we mainly construct multiple spanning paths in hypercube with faulty vertices and edges.

An n-dimensional hypercube $Q_n(V_b \cup V_w, E)$ is a bipartite graph whose vertices are labeled by distinct n-bit binary strings. Two vertices are linked by an edge if and only if their labels differ exactly in one bit. An i-edge (x, y) is an edge that x and y differ in the i-th bit. The hypercube Q_n can be constructed recursively as $Q_n = Q_{n-1} \times K_2$. We can partition Q_n into two subgraphs Q^0_{n-1} and Q^1_{n-1} by choosing any one bit of binary string.

Let V_b be the black vertex set and V_w white vertex set of Q_n. We denote the black and white vertex set of Q^j_{n-1} with V^j_b and V^j_w, for $j = 0, 1$. And let $V^j = V^j_b \cup V^j_w$ for $j = 0, 1$. Thus, $V_b = V^0_b \cup V^1_b, V_w = V^0_w \cup V^1_w, V = V_b \cup V_w = V^0 \cup V^1$.

Let F_b be the set of black faulty vertices of Q_n and F_w the set of white faulty vertices of Q_n, F_e the set of faulty edges of Q_n. Similarly, we also use F^j_b and F^j_w and F^j_e to denote the black and white faulty vertex set and faulty edge set of Q^j_{n-1}, respectively, for $j = 0, 1$. Thus, $F_b = F^0_b \cup F^1_b, F_w = F^0_w \cup F^1_w, F^0 = F^0_b \cup F^0_w \cup F^0_e, F^1 = F^1_b \cup F^1_w \cup F^1_e, F = F_b \cup F_w \cup F_e = F^0 \cup F^1$.

In [1], Caha et al. proposed the multiple spanning disjoint paths problem for hypercube. Let s_i, t_i, for $1 \leq i \leq k$, be vertices of Q_n. The $\{s_i, t_i\}^k_{i=1}$ is a *connectable family* if there exists k spanning paths of Q_n between s_i and t_i, for $1 \leq i \leq k$. The $\{s_i, t_i\}^k_{i=1}$ is *balanced* if it has the same number of vertices in each partite set. Caha showed that every balanced family $\{s_i, t_i\}^n_{i=1}$ is connectable in Q_{2n} if the distance of every pair s_i, t_i is odd. Caha also showed that every balanced family $\{s_i, t_i\}^n_{i=1}$ is connectable in Q_{6n}.

In [5], the authors presented the vertex fault tolerance for multiple spanning disjoint paths in hypercube. Let $\{s_i, t_i\}^{F_b, K_b}_{F_w, K_w}$ be a family of $G = (V_b \cup V_w, E)$ where $K_b(\subset V_b) \cup K_w(\subset V_w) = \{s_i, t_i | 1 \leq i \leq \frac{|K_b|+|K_w|}{2}\}$ is the set of fault-free end vertices, $F_b \subset V_b$ and $F_w \subset V_w$ are sets of faulty vertices. The family $\{s_i, t_i\}_{F_w, K_w}^{F_b, K_b}$ is balanced if $|K_w| + 2|F_w| = |K_b| + 2|F_b|$. The family $\{s_i, t_i\}_{F_w, K_w}^{F_b, K_b}$ is connectable if there exist $(|K_b| + |K_w|)/2$ spanning paths $P(s_i, t_i)$, for $1 \leq i \leq (|K_b| + |K_w|)/2$, in $G - F_b - F_w$.

We also use K_b^j and K_w^j to denote the set of black and white end vertices of Q_{n-1}^j, respectively, for $j = 0, 1$. Let $K^j = K_b^j \cup K_w^j$ for $j = 0, 1$ and $K = K^0 \cup K^1$. Thus, $K_b = K_b^0 \cup K_b^1$ and $K_w = K_w^0 \cup K_w^1$. Let $K_w^{01} = \{v_w | v_w \in K_w^0$ and $u \in K^1$, for $\langle v_w, u\rangle$ is a pair of $K.\}$. Let $K_b^{01} = \{v_b | v_b \in K_b^0$ and $u \in K^1$, for $\langle v_b, u\rangle$ is a pair of $K.\}$. Let v be a vertex of V^0 and U be a vertex subset of V^0. We use $\phi(v)$ to denote the neighbor of v in V^1. We further let $\phi(U) = \{\phi(v) | v \in U \subseteq V^0\}$.

The following lemma is proved in [5].

Lemma 2 *Every balanced family* $\{s_i, t_i\}_{F_w, K_w}^{F_b, K_b}$ *of hypercube* Q_n *is connectable if* $|F_b| + |F_w| + |K_b| + |K_w| \leq n, 4|F_b| + 2|K_b| = 4|F_w| + 2|K_w| \leq n + 1$, *for* $n \geq 3$.

In the following, we will investigate the vertex and edge fault tolerance for multiple spanning disjoint paths in hypercube. We will prove the following theorem.

Theorem 2 *Every balanced family* $\{s_i, t_i\}_{F_w, K_w}^{F_b, K_b}$ *of* $Q_n - F_e$ *is connectable if* $|F_b| + |F_w| + |K_b| + |K_w| + |F_e| \leq n, 4|F_b| + 2|K_b| + |F_e| = 4|F_w| + 2|K_w| + |F_e| \leq (n+1)$, *for* $n \geq 3$.

Proof:

When $|F_e| = 0$, applying Lemma 2, we can obtain this theorem is hold. In the following, we will assume that $|F_e| \geq 1$. We will prove this theorem by induction on n. For $n \geq 3$, we assume that every balanced family $\{s_i, t_i\}_{F_w, K_w}^{F_b, K_b}$ of hypercube $Q_{n-1} - F_e$ is connectable where $|F_b| + |F_w| + |K_b| + |K_w| + |F_e| \leq (n-1)$, $4|F_b| + 2|K_b| + |F_e| \leq n = 4|F_w| + 2|K_w| + |F_e| \leq n$. Since Q_3 is 1 edge Hamiltonian laceable [7], this theorem is holds for $n = 3$. We will partition Q_n into two subgraphs Q_{n-1}^0 and Q_{n-1}^1 with a bit i which some faulty edge is i-edge. Thus, $|F_e^j| \leq |F_e| - 1$ for $j = 0, 1$. Without loss of generality, we can assume that $|F_b| \geq |F_w|$. Thus, $|K_w| \geq |K_b|$.

Case 1: $|F^0| + |K^0| = 0$ or $|F^1| + |K^1| = 0$

Without loss of generality, we can assume that $F_b \cup F_w \cup K \in Q_{n-1}^0$. Since $|F_b^0| + |F_w^0| + |K_b^0| + |K_w^0| + |F_e| - 1 \leq (n-1), 4|F_b^0| + 2|K_b^0| + |F_e| - 1 \leq n = 4|F_w^0| + 2|K_w^0| + |F_e| - 1 \leq n$, $\{s_i, t_i\}_{F_w^0, K_w^0}^{F_b^0, K_b^0}$ is connectable family of $Q_{n-1}^0 - F_e^0$. Therefore, we can construct $\frac{|K_b^0| + |K_w^0|}{2}$ spanning paths of $Q_{n-1}^0 - F_e^0$. One of these paths is $\langle s_1, ..., x, y, ..., t_1\rangle$. There is a Hamiltonian path $\langle \phi(x), ..., \phi(y)\rangle$ of $Q_{n-1}^1 - F_e^1$. Thus, we can construct $\frac{|K_b| + |K_w|}{2}$ spanning paths of $Q_n - F_e$, as illustrated in Figure 4.(a).

Case 2: $|F^0| + |K^0| \geq 1$ and $|F^1| + |K^1| \geq 1$

Let $U_b^0 \subset (V_b^0 - F_b^0 - K_b^0)$ with $\phi(U_b^0) \subset (V_w^1 - F_w^1 - K_w^1)$, $|U_b^0| = \max(|K_w^{01}|, (2|F_w^0| + |K_w^0|) - (2|F_b^0| + |K_b^0|))$ and $U_w^0 \subset (V_w^0 - F_w^0 - K_w^0)$ with $\phi(U_w^0) \subset (V_b^1 - F_b^1 - K_b^1)$, $|U_w^0| = \max(|K_b^{01}|, (2|F_b^0| + |K_b^0|) + |U_b^0| - (2|F_w^0| + |K_w^0|))$.

Since $|F_b^0| + |F_w^0| + |K_b^0| + |U_b^0| + |K_w^0| + |U_w^0| \leq (n-1)$, $4|F_b^0| + 2(|K_b^0| + |U_b^0|) = 4|F_w^0| + 2(|K_w^0| + |U_w^0|) \leq n$, $\{s_i, t_i\}_{F_w^0, K_w^0 \cup U_w^0}^{F_b^0, K_b^0 \cup U_b^0}$ is connectable family of Q_{n-1}^0, we can construct $\frac{|K_b^0| + |U_b^0| + |K_w^0| + |U_w^0|}{2}$ spanning paths of Q_{n-1}^0. Because of $|F_b^1| + |F_w^1| + |K_b^1| + |\phi(U_w^0)| + |K_w^1| + |\phi(U_b^0)| \leq (n-1)$, $4|F_b^1| + 2(|K_b^1| + |\phi(U_w^0)|) = 4|F_w^1| + 2(|K_w^1| + |\phi(U_b^0)|) \leq n$, $\{s_i, t_i\}_{F_w^1, K_w^0 \cup \phi(U_b^0)}^{F_b^1, K_b^0 \cup \phi(U_w^0)}$ is connectable family of Q_{n-1}^1. There exist $\frac{|K_b^1| + |U_b^0| + |K_w^1| + |U_w^0|}{2}$ spanning paths of Q_{n-1}^1. Therefore, we can construct $\frac{|K_b| + |K_w|}{2}$ spanning paths in Q_n, as illustrated in Figure 4.(b). □

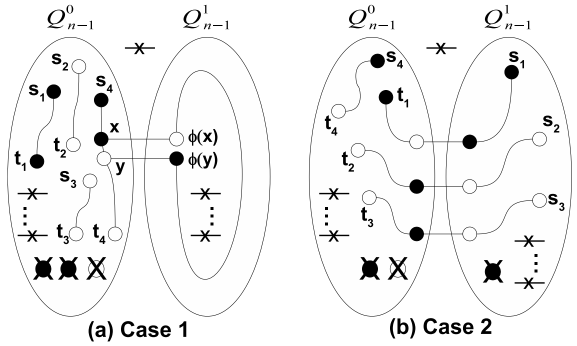

Figure 4: Illustration of **Theorem 2**.

4 Fault Hamiltonicity for node expansion of hypercube

In [4], the authors defined the *t-node expansion* operation as follows. Let x be a vertex of graph $G = (V, E)$ with $d_G(x) = t$. Let $\{x_1, x_2, \cdots, x_t\}$ be the set of neighbor of x. The t-node expansion $X(G, x)$ of G on x is the graph obtained from G by replacing x with a complete graph K_t. Let $V(K_t) = \{k_1, k_2, \cdots, k_t\}$. That is, $V(X(G, x)) = V - \{x\} \cup \{k_1, k_2, \cdots, k_t\}$ and $E(X(G, x)) = E \cup E(K_t) \cup \{(x_i, k_i) | 1 \leq i \leq t\} - \{(x, x_i) | 1 \leq i \leq t\}$. Moreover, the node expansion can be applied on a vertex subset. The node expansion of $G = (V, E)$ on the subset $U \subseteq V$, denoted by $X(G, U)$, is the graph that is obtained from G by a sequence node expansion operations on every node $u \in U$. Let $N_G(x) = \{(x, x_i)|$ for all $1 \leq i \leq t\}$ and $M_{X(G,U)}(x) = V(K_t) \cup E(K_t) \cup \{(k_i, x_i)|$ for all $1 \leq i \leq t\}$ for $x \in U$. The graph G and $X(G, x)$ are illustrated in Figure 5.

Let F be the set of faulty vertices and faulty edges of $X(G,U)$ and $F_X(x) = F \cap M_{X(G,U)}(x)$ for $x \in U$. Let $V' \subseteq V(K_t - F_X(x))$ be the set such that every m pairs of vertices in V' there exist m spanning disjoint paths of $K_t - F_X(x)$ for $1 \leq m \leq \lfloor \frac{n-|F_X(x)|}{2} \rfloor$ and $|V'| = n - |F_X(x)|$. Let $K_{X(G,U)-F(x)} = \{x_i | (x, x_i) \in E(G)$ and $k_i \in V'$ and $(x_i, k_i) \notin F\}$. Let $F^e_X(x) = \{(x_i, k_i) |$ for $k_i \notin V'$ or $(x_i, k_i) \in F$ and $1 \leq i \leq t\}$. Let $F^e_G(x) = \{(x, x_i) |$ for $(x_i, k_i) \in F^e_X(x)\}$. Thus $|F^e_X(x)| = |F^e_G(x)| = |F_X(x)|$.The following lemma is proved in [4].

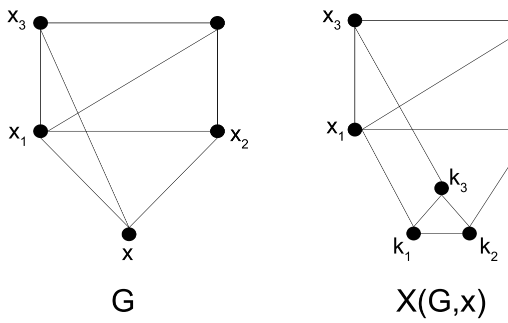

Figure 5: Illustration of **node expansion**.

Lemma 3 *Given $F_1 \subset (V(G-x) \cup E(G-x))$. If we delete any f edges of $N_G(x)$ from the graph $G - F_1$ such that the remaining graph is Hamiltonian for $f \leq t-2$, then the graph $X(G,x) - (F_1 \cup F_3)$ is Hamiltonian, where F_3 is a subset of $M_{X(G,x)}(x)$ and $|F_3| = f$.*

Let $X(Q_n, \{x_b, x_w\})$ be the n-node expansion of $Q_n = (V_b \cup V_w, E)$ on $\{x_b, x_w\}$ for $x_b \in V_b$ and $x_w \in V_w$. Let K^b_n and K^w_n be the complete graphs replacing x_b and x_w, respectively. That is, $V(X(Q_n, \{x_b, x_w\})) = V_b \cup V_w \cup V(K^b_n) \cup V(K^w_n) - \{x_b\} - \{x_w\}$. Let F be the set of faulty element of $X(Q_n, \{x_b, x_w\})$. Let $F_b = F \cap V_b$, $F_w = F \cap V_w$, $F' = F - F_b - F_w$ and $f_{max} = \max(|F_b|, |F_w|)$. Let $F_X(x_w) = F' \cap M_{X(Q_n,\{x_b,x_w\})}(x_w)$ and $F_X(x_b) = F' \cap M_{X(Q_n,\{x_b,x_w\})}(x_b)$. Applying the definition of $F^e_X(x)$, we can define that $F^e_{Q_n}(x_b) = \{(x_b, x^i_b) |$ for $(x^i_b, k^i_b) \in F^e_X(x_b)\}$ and $F^e_{Q_n}(x_w) = \{(x_w, x^i_w) |$ for $(x^i_w, k^i_w) \in F^e_X(x_w)\}$. We also use F^e_X to denote $F' - F_X(x_b) - F_X(x_w)$. We can prove the following theorem.

Theorem 3 *The graph $X(Q_n, \{x_b, x_w\}) - F$ is Hamiltonian if*

1. $|F_b| = |F_w| = 0, |F'| \leq n-2,$
2. $0 < |F_b| = |F_w| \leq \lceil \frac{n}{4} \rceil - 1, |F'| \leq n - 1 - 4|F_b|,$
3. $0 \leq |F_w| \neq |F_b| \leq \lceil \frac{n}{4} \rceil - 2, |F'| \leq n - 3 - 4f_{max}.$

Proof:

The graph K^b_n is the complete graph replacing x_b in $X(Q_n, \{x_b, x_w\})$. Let $F_{K^b_n} = F \cap (V(K^b_n) \cup E(K^b_n))$ and $F_{K^w_n} = F \cap (V(K^w_n) \cup E(K^w_n))$. Let $F^e_{Q_n} = F^e_X \cup F^e_{Q_n}(x_w) \cup F^e_{Q_n}(x_b)$. Thus $|F^e_{Q_n}| = |F^e_X| + |F^e_{Q_n}(x_w)| + |F^e_{Q_n}(x_b)| = |F'| - |F_X(x_b)| - |F_X(x_w)| + |F^e_{Q_n}(x_w)| + |F^e_{Q_n}(x_b)| = |F'|$. We will prove this theorem by the following cases.

Case 1: $|F_b| = |F_w| = 0$.
Thus $|F'| \leq n-2$. Since $F = F' = F_X(x_b) \cup F_X(x_w) \cup F^e_X$, $|F_X(x_b)| + |F_X(x_w)| + |F^e_X| \leq n-2$. When we delete any $|F_X(x_b)|$ edges of $N_{Q_n}(x_b)$ and $|F_X(x_w)|$ edges of $N_{Q_n}(x_w)$ from $Q_n - F^e_X$, the remaining graph is Hamiltonian since Q_n is $(n-2)$-edge Hamiltonian. Applying Lemma 3, $X(Q_n, \{x_b, x_w\}) - F^e_X(x_b) - F^e_X(x_w)$ is also Hamiltonian.

Case 2: $|F_b| = |F_w| > 0$.
Thus $|F'| \leq n - 4|F_b|$. Let $F^*_{Q_n}(x_b)$ be the set of arbitrary $|F_X(x_b)|$ edges adjacent to x_b of Q_n and $F^*_{Q_n}(x_w)$ be the set of arbitrary $|F_X(x_w)|$ edges adjacent to x_w of Q_n. We also denote the set $F^*_{Q_n}(x_b) \cup F^*_{Q_n}(x_w) \cup F^e_X$ by $F^*_{Q_n}$. Since $|F_b| + |F_w| + 2 + |F^*_{Q_n}| \leq 2|F_b| + 2 + n - 1 - 4|F_b| = n + 1 - 2|F_b| \leq n-1$, $4|F_b| + 2 + |F^*_{Q_n}| = 4|F_w| + 2 + |F^*_{Q_n}| \leq 4|F_b| + 2 + n - 1 - 4|F_b| \leq n+1$, there exists a Hamiltonian path of $Q_n - F_b - F_w - F^*_{Q_n}$ between every pair of vertices with odd distance. Thus, $Q_n - F_b - F_w - F^*_{Q_n}$ is Hamiltonian laceable. This graph is also Hamiltonian. Applying the definition of $F^e_{Q_n}$, we can know that $|F^e_{Q_n}| \leq |F^*_{Q_n}|$. Thus $Q_n - F_b - F_w - F^e_{Q_n}$ is Hamiltonian. Applying Lemma 3, $X(Q_n, \{x_b, x_w\}) - F_b - F_w - F'$ is also Hamiltonian.

Case 3: $|F_b| \neq |F_w|$. Thus $|F'| \leq n - 3 - 4f_{max}$.
Without loss of generality, we can assume that $|F_b| \geq |F_w|$. Thus, $f_{max} = |F_b|$. Since $F_{K^b_n} \subseteq F_X(x_b) \subseteq F'$, $|F_{K^b_n}| \leq |F'| \leq n - 3 - 4|F_b| \leq n-2$. Applying Theorem 1, we can obtain a set $V' \subseteq (V(K^b_n) - F_{K^b_n})$ with $|V'| = n - |F_{K^b_n}|$, such that any m pairs of vertices in $|V'|$, there exist m spanning disjoint paths of $K^b_n - F_{K^b_n}$ for $1 \leq m \leq \lfloor \frac{n-|F_{K^b_n}|}{2} \rfloor$. Since $2|F_b| + 2 - 2|F_w| \leq 4|F_b| + 3 \leq n - |F_X(x_b)|$, we will construct $|F_b| + 1 - |F_w|$ spanning disjoint paths $P(k_{s_i}, k_{t_i})$ of $K^b_n - F_{K^b_n}$ for $(k_{s_i}, x_{s_i}) \notin F_X(x_b)$ and $(k_{t_i}, x_{t_i}) \notin F_X(x_b)$, $1 \leq i \leq |F_b| + 1 - |F_w|$.
Let $F'_b = \{x_b\} \cup F_b$ and $K_w \subset K_{X(Q_n,\{x_b,x_w\})-F}(x_b)$ with $|K_w| = 2(|F'_b| - |F_w|)$ and $K_w \cap F_w = \emptyset$. Hence $|F'_b| + |F_w| + |K_w| + |F^e_{Q_n}| = 3|F_b| + 3 - |F_w| + n - 3 - 4|F_b| = n - |F_b| - |F_w| < n$, $4|F'_b| + |F^e_{Q_n}| = 4|F_w| + 2|K_w| + |F^e_{Q_n}| = 4|F_b| + 4 + n - 3 - 4|F_b| \leq n+1$. Applying Theorem 2, we can obtain that for any $|K_w|$ vertices there exist $\frac{|K_w|}{2}$ spanning disjoint paths of $Q_n - F'_b - F_w - F^e_{Q_n}$ between every pair of vertices of K_w. We can construct $\frac{|K_w|}{2}$ spanning disjoint paths $P(x_{t_1}, x_{s_2}), P(x_{t_2}, x_{s_3}), \cdots, P(x_{t_{\frac{|K_w|}{2}}}, x_{s_1})$ of

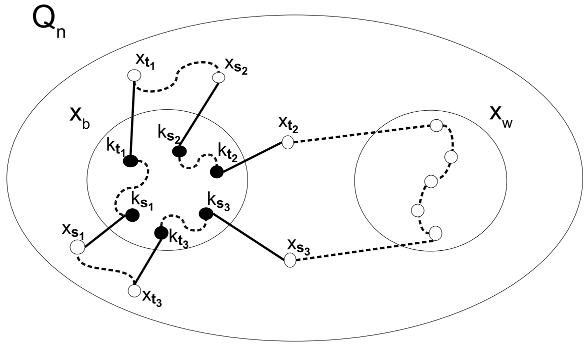

Figure 6: Illustration of **Theorem 3**.

$Q_n - F'_b - F_w - F^e_{Q_n}$. Therefore, $\langle k_{s_1} \rightarrow P(k_{s_1}, k_{t_1}) \rightarrow k_{t_1}, x_{t_1} \rightarrow P(x_{t_1}, x_{s_2}) \rightarrow x_{s_2}, k_{s_2} \rightarrow \cdots \rightarrow P(x_{t_{\frac{\lfloor K_w \rfloor}{2}}}, x_{s_1}) \rightarrow x_{s_1}, k_{s_1}\rangle$ forms a Hamiltonian cycle of $X(Q_n, x_b) - F_b - F_w - F^e_X - F_X(x_b) - F_{Q_n}(x_w)$. Thus, $X(Q_n, x_b) - F_b - F_w - F^e_X - F_X(x_b) - F_{Q_n}(x_w)$ is Hamiltonian, as illustrated in Figure 6. Applying Lemma 3, we can obtain that $X(Q_n, \{x_b, x_w\}) - F_b - F_w - F^e_X - F_X(x_b) - F_X(x_w) = X(Q_n, \{x_b, x_w\}) - F$ is Hamiltonian since $|F_X(x_w)| = |F_{Q_n}(x_w)|$. □

Corollary 1 *Let $X(Q_n, \{x_b, x_w\})$ is k-Hamiltonian where x_b and x_w are two vertices in Q_n with odd distance for $k = \lceil \frac{n}{4} \rceil - 2$.*

5 Conclusion

In this paper, we prove first the fault tolerance for multi-spanning disjoint paths in complete graph K_n. When $F \subset (V \cup E)$ is a faulty set with $|F| \leq n - 2$, we show that there exist m spanning disjoint paths in $K_n - F$ for $1 \leq m \leq \lfloor \frac{n-|F|}{2} \rfloor$. Secondly, we discuss the fault tolerance for balanced and connectable property of hypercube Q_n. We show that Q_n is balanced and connectable if $|F_b| + |F_w| + |K_b| + |K_w| + |F_e| \leq n$,$4|F_b| + 2|K_b| + |F_e| \leq n+1$ and $4|F_w| + 2|F_b| + |F_e| \leq n+1$, for $n \geq 3$. Applying these results described above, we construct the variant of hypercube $X(Q_n, \{x_b, x_w\})$ with node expansion on one black node x_b and one white node x_w of hypercube. We prove that $X(Q_n, \{x_b, x_w\}) - (F_b \cup F_w \cup F')$ is Hamiltonian if

1. $|F_b| = |F_w| = 0, |F'| \leq n - 2$.
2. $0 < |F_b| = |F_w| \leq \lceil \frac{n}{4} \rceil - 1, |F'| \leq n - 1 - 4|F_b|$.
3. $0 \leq |F_w| \neq |F_b| \leq \lceil \frac{n}{4} \rceil - 2, |F'| \leq n - 3 - 4f_{max}$, for $f_{max} = \max\{|F_b|, |F_w|\}$.

Thus, we derive that $X(Q_n, x_b, x_w)$ is k-Hamiltonian for $k = \lceil \frac{n}{4} \rceil - 2$.

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