Two Spanning Disjoint Paths with Required Length in Augmented Cubes

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Abstract

In this article, we introduce 2RP-property in the augmented cube AQ_n: Let $\{\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}\}$ be any four distinct vertices of AQ_n. Let l_1 and l_2 be two integers with $l_1 \geq d_{AQ_n}(\mathbf{u}, \mathbf{v})$, $l_2 \geq d_{AQ_n}(\mathbf{x}, \mathbf{y})$, and $l_1 + l_2 = 2^n - 2$. Then there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{u}$ and $\mathbf{v}$ with $l(P_1) = l_1$, (2) P_2 is a path joining $\mathbf{x}$ and $\mathbf{y}$ with $l(P_2) = l_2$, and (3) $P_1 \cup P_2$ spans AQ_n except some special conditions.

Keywords: hamiltonian, augmented cubes.

1 Introduction

Interconnection networks play an important role in parallel computing/communication systems. The graph embedding problem is a central issue in evaluating a network. The graph embedding problem asked if the quest graph is a subgraph of a host graph, and an important benefit of the graph embeddings is that we can apply existing algorithm for guest graphs to host graphs. This problem has attracted a burst of studies in recent years. Cycle networks and path networks are suitable for designing simple algorithms with low communication costs. The cycle embedding problem, which deals with all possible lengths of the cycles in a given graph, is investigated in a lot of interconnection networks [6, 10, 12, 13]. The path embedding problem, which deals with all possible lengths of the paths between given two vertices in a given graph, is investigated in a lot of interconnection networks [3–5, 12–14].

In this article, a network is represented as a loopless undirected graph. For the graph definitions and notation, we follow [1]. Let $G = (V, E)$ be a graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$. We use $Nbd_G(u)$ to denote the set $\{v \mid (u, v) \in E(G)\}$. The *degree* of a vertex u in G, denoted by $\deg_G(u)$, is $|Nbd_G(u)|$. We use $\delta(G)$ to denote $\min\{\deg_G(u) \mid u \in V(G)\}$. A graph is k*-regular* if $\deg_G(u) = k$ for every vertex u in G. A path is a sequence of adjacent vertices, written as $\langle v_0, v_1, \ldots, v_m\rangle$, in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct except that possibly $v_0 = v_m$. We also write the path $\langle v_0, P, v_m\rangle$, where $P = \langle v_0, v_1, \ldots, v_m\rangle$. The *length* of a path P, denoted by $l(P)$, is the number of edges in P. Let u and v be two vertices of G. The *distance* between u and v denoted by $d_G(u, v)$ is the length of the shortest path of G joining u and v. The *diameter* of a graph G, denoted by $D(G)$, is $\max\{d_G(u, v) \mid u, v \in V(G)\}$. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A *hamiltonian cycle* is a cycle of length $V(G)$. A *hamiltonian path* is a path of length $V(G) - 1$.

The hypercube Q_n is one of the most popular interconnection networks for parallel computer/comminication system [11]. This is partly due to its attractive properties, such as regularity, recursive structure, vertex and edge symmetry, maximum connectivity, as well as effective routing and broadcasting algorithm. The augmented cube AQ_n is a variation of Q_n, proposed by Choudum and Sunitha [2], and not only retains some favorable properties of Q_n but also processes some embedding properties that Q_n does not [2, 7–9, 13]. For example, AQ_n contains cycles of all lengths from 3 to 2^n, but Q_n contains only even cycles.

For the path embedding problem on the augmented cube, Ma et al. [13] proved that between any two distinct vertices $\mathbf{x}$ and $\mathbf{y}$ of AQ_n, there exists a path $P_l(\mathbf{x}, \mathbf{y})$ of length l with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \leq l \leq 2^n - 1$. Obviously, we expect that such a path $P_l(\mathbf{x}, \mathbf{y})$ can be further extended by including the vertices not in $P_l(\mathbf{x}, \mathbf{y})$ into a hamiltonian path from $\mathbf{x}$ to a fixed vertex $\mathbf{z}$ or a hamiltonian cycle. For this reason, we prove that for any three distinct vertices $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ of AQ_n, and for any $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \leq l \leq 2^n - 1 - d_{AQ_n}(\mathbf{y}, \mathbf{z})$ there exists a hamiltonian path $R(\mathbf{x}, \mathbf{y}, \mathbf{z}; l)$ from $\mathbf{x}$ to $\mathbf{z}$ such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x}, \mathbf{y}) = l$. As a corollary, we prove that for any two distinct vertices $\mathbf{x}$ and $\mathbf{y}$, and for any $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \leq l \leq 2^{n-1}$, there exists a hamiltonian cycle $S(\mathbf{x}, \mathbf{y}; l)$ such that $d_{S(\mathbf{x},\mathbf{y};l)}(\mathbf{x}, \mathbf{y}) = l$.

In the following section, we introduce the definition and some properties of the augmented cubes. In Section 3, we introduce the 2RP-property for the augmented cube AQ_n and prove that AQ_n satisfies the 2RP-property if $n \geq 2$. We make some remarks to illustrate that some interesting properties of augmented cubes are consequences of 2RP-property in the final section.

2 Preliminaries

In this section, we introduce some properties of the augmented cubes. Assume that $n \geq 1$ is an integer. The graph of the n*-dimensional augmented cube*, denoted by AQ_n, has 2^n vertices, each labeled by an n-bit binary string $V(AQ_n) = \{u_1u_2 \ldots u_n \mid u_i \in \{0, 1\}\}$. For $n = 1$, AQ_1 is the graph K_2 with vertex set $\{0, 1\}$. For $n \geq 2$, AQ_n can be recursively constructed by two copies of AQ_{n-1}, denoted by AQ^0_{n-1} and AQ^1_{n-1}, and by adding 2^n edges between AQ^0_{n-1} and AQ^1_{n-1} as follows:

Let $V(AQ^0_{n-1}) = \{0u_2u_3 \dots u_n \mid u_i = 0 \text{ or } 1 \text{ for } 2 \leq i \leq n\}$ and $V(AQ^1_{n-1}) = \{1v_2v_3 \dots v_n \mid v_i = 0 \text{ or } 1 \text{ for } 2 \leq i \leq n\}$. A vertex $\mathbf{u} = 0u_2u_3 \dots u_n$ of AQ^0_{n-1} is adjacent to a vertex $\mathbf{v} = 1v_2v_3 \dots v_n$ of AQ^1_{n-1} if and only if one of the following cases holds.

(i) $u_i = v_i$, for $2 \leq i \leq n$. In this case, $(\mathbf{u}, \mathbf{v})$ is called a *hypercube edge*. We set $\mathbf{v} = \mathbf{u}^h$.

(ii) $u_i = \bar{v}_i$, for $2 \leq i \leq n$. In this case, $(\mathbf{u}, \mathbf{v})$ is called a *complement edge*. We set $\mathbf{v} = \mathbf{u}^c$.

The augmented cubes AQ_1, AQ_2, AQ_3 and AQ_4 are illustrated in Figure 1. It is proved in [2] that AQ_n is a vertex transitive, $(2n-1)$-regular, and $(2n-1)$-connected graph with 2^n vertices for any positive integer n. Let i be any index with $1 \leq i \leq n$ and $\mathbf{u} = u_1u_2u_3 \dots u_n$ be a vertex of AQ_n. We use $\mathbf{u}^i$ to denote the vertex $\mathbf{v} = v_1v_2v_3 \dots v_n$ such that $u_j = v_j$ with $1 \leq j \neq i \leq n$ and $u_i = \bar{v}_i$. Moreover, we use $\mathbf{u}^{i*}$ to denote the vertex $\mathbf{v} = v_1v_2v_3 \dots v_n$ such that $u_j = v_i$ for $j < i$ and $u_j = \bar{v}_j$ for $i \leq j \leq n$. Obviously, $\mathbf{u}^n = \mathbf{u}^{n*}$, $\mathbf{u}^1 = \mathbf{u}^h$, $\mathbf{u}^c = \mathbf{u}^{1*}$, and $Nbd_{AQ_n}(\mathbf{u}) = \{\mathbf{u}^i \mid 1 \leq i \leq n\} \cup \{\mathbf{u}^{i*} \mid 1 \leq i < n\}$.

Lemma 1. *Assume that* $n \geq 2$. *Then* $|Nbd_{AQ_n}(\mathbf{u}) \cap Nbd_{AQ_n}(\mathbf{v})| \geq 2$ *if* $(\mathbf{u}, \mathbf{v}) \in E(G)$.

Proof. We prove this lemma by induction. Since AQ_2 is isomorphic to the complete graph K_4, the lemma holds for $n = 2$. Assume the lemma holds for $2 \leq k < n$. Suppose that $\{\mathbf{u}, \mathbf{v}\} \subset V(AQ^i_{n-1})$ for some $i \in \{0, 1\}$. By induction, $|Nbd_{AQ_n}(\mathbf{u}) \cap Nbd_{AQ_n}(\mathbf{v})| \geq 2$. Thus, consider the case that either $\mathbf{v} = \mathbf{u}^h$ or $\mathbf{v} = \mathbf{u}^c$. Obviously, $\{\mathbf{u}^{2*}, \mathbf{u}^c\} \subset Nbd_{AQ_n}(\mathbf{u}) \cap Nbd_{AQ_n}(\mathbf{v})$ if $\mathbf{v} = \mathbf{u}^h$; and $\{\mathbf{u}^{2*}, \mathbf{u}^h\} \subset Nbd_{AQ_n}(\mathbf{u}) \cap Nbd_{AQ_n}(\mathbf{v})$ if $\mathbf{v} = \mathbf{u}^c$. Then the statement holds. □

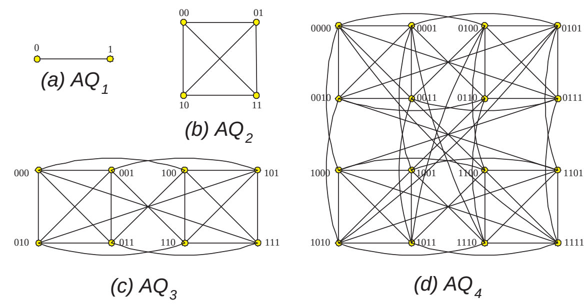

Figure 1: The augmented cubes AQ_1, AQ_2, AQ_3 and AQ_4.

The following lemma can easily be obtained from the definition of AQ_n.

Lemma 2. *Assume that* $n \geq 3$. *For any two different vertices* $\mathbf{u}$ *and* $\mathbf{v}$ *of* AQ_n, *there exists two other vertices* $\mathbf{x}$ *and* $\mathbf{y}$ *of* AQ_n *such that the subgraph of* $\{\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}\}$ *containing a four cycle.*

Lemma 3. *[8] Let* F *be a subset of* $V(AQ_n)$. *Then there exists a hamiltonian path between any two vertices of* $V(AQ_n) - F$ *if* $|F| \leq 2n - 4$ *for* $n \geq 4$ *and* $|F| \leq 1$ *for* $n = 3$.

Lemma 4. *[2] Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be any two vertices in* AQ_n *with* $n \geq 2$. *Suppose that both* $\mathbf{u}$ *and* $\mathbf{v}$ *are in* AQ^i_{n-1} *for* $i = 0, 1$. *Then* $d_{AQ_n}(\mathbf{u}, \mathbf{v}) = d_{AQ^i_{n-1}}(\mathbf{u}, \mathbf{v})$. *Suppose that* $\mathbf{u}$ *is a vertex in* AQ^i_{n-1} *and* $\mathbf{v}$ *is a vertex in* AQ^{1-i}_{n-1}. *Then there exist two shortest paths* P_1 *and* P_2 *of* AQ_n *joining* $\mathbf{u}$ *to* $\mathbf{v}$ *such that* $(V(P_1) - \{\mathbf{v}\}) \subset V(AQ^i_{n-1})$ *and* $(V(P_2) - \{\mathbf{u}\}) \subset V(AQ^{1-i}_{n-1})$.

With Lemma 4, we have the following corollary.

Corollary 5. *Assume that* $n \geq 3$. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two vertices of* AQ_n *with* $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \geq 2$. *Then, there are* *two vertices* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $Nbd_{AQ_n}(\mathbf{x})$ *with* $d_{AQ_n}(\mathbf{p},\mathbf{y}) = d_{AQ_n}(\mathbf{q},\mathbf{y}) = d_{AQ_n}(\mathbf{x},\mathbf{y}) - 1$.

Lemma 6. *[8] Let* $\{\mathbf{u},\mathbf{v},\mathbf{x},\mathbf{y}\}$ *be any four distinct vertices of* AQ_n *with* $n \geq 2$. *Then there exist two disjoint paths* P_1 *and* P_2 *such that (1)* P_1 *is a path joining* $\mathbf{u}$ *and* $\mathbf{v}$, *(2)* P_2 *is a path joining* $\mathbf{x}$ *and* $\mathbf{y}$, *and (3)* $P_1 \cup P_2$ *spans* AQ_n.

We refer to Lemma 6 as 2P-property of the augmented cube. This property is used for many applications of the augmented cubes [7,8]. Obviously, $l(P_1) \geq d_{AQ_n}(\mathbf{u},\mathbf{v})$ and $l(P_2) \geq d_{AQ_n}(\mathbf{x},\mathbf{y})$, and $l(P_1) + l(P_2) = 2^n - 2$. We expect that $l(P_1)$, hence, $l(P_2)$ can be an arbitrarily integer with the above constraint. However, such expectation is almost true. Let us consider AQ_3. Suppose that $\mathbf{u} = 001$, $\mathbf{v} = 110$, $\mathbf{x} = 101$, and $\mathbf{y} = 010$. Thus, $d_{AQ_3}(\mathbf{u},\mathbf{v}) = 1$ and $d_{AQ_3}(\mathbf{x},\mathbf{y}) = 1$. We can find P_1 and P_2 with $l(P_1) \in \{1,3,5\}$. Note that $\{\mathbf{x},\mathbf{y}\} = Nbd_{AQ_3}(\mathbf{u}) \cap Nbd_{AQ_3}(\mathbf{v})$. We can not find P_1 with $l(P_1) = 2$. Again, $\{\mathbf{u},\mathbf{v}\} = Nbd_{AQ_3}(\mathbf{x}) \cap Nbd_{AQ_3}(\mathbf{y})$. We can not find P_2 with $l(P_2) = 2$. Hence, we cannot find P_1 with $l(P_1) = 4$. Similarly, we consider AQ_4. Suppose that $\mathbf{u} = 0000$, $\mathbf{v} = 1001$, $\mathbf{x} = 0001$ and $\mathbf{y} = 1000$. Thus, $d_{AQ_4}(\mathbf{u},\mathbf{v}) = 2$ and $d_{AQ_4}(\mathbf{x},\mathbf{y}) = 2$. We can find P_1 and P_2 with $l(P_1) \in \{3,4,\ldots,11\}$. Note that $\{\mathbf{x},\mathbf{y}\} = Nbd_{AQ_4}(\mathbf{u}) \cap Nbd_{AQ_4}(\mathbf{v})$. We can not find P_1 with $l(P_1) = 2$. Again, $\{\mathbf{u},\mathbf{v}\} = Nbd_{AQ_4}(\mathbf{x}) \cap Nbd_{AQ_4}(\mathbf{y})$. We can not find P_2 with $l(P_2) = 2$.

3 The 2RP-property of the augmented cubes

In this section, we introduce the 2RP-property for the augmented cube AQ_n and prove that AQ_n satisfies the 2RP-property if $n \geq 2$. First, we propose the 2RP-property of AQ_n with $n \geq 2$: *Let* $\{\mathbf{u},\mathbf{v},\mathbf{x},\mathbf{y}\}$ *be any four distinct vertices of* AQ_n. *Let* l_1 *and* l_2 *be two integers with* $l_1 \geq d_{AQ_n}(\mathbf{u},\mathbf{v})$, $l_2 \geq d_{AQ_n}(\mathbf{x},\mathbf{y})$, *and* $l_1 + l_2 = 2^n - 2$. *Then there exist two disjoint paths* P_1 *and* P_2 *such that (1)* P_1 *is a path joining* $\mathbf{u}$ *and* $\mathbf{v}$ *with* $l(P_1) = l_1$, *(2)* P_2 *is a path joining* $\mathbf{x}$ *and* $\mathbf{y}$ *with* $l(P_2) = l_2$, *and (3)* $P_1 \cup P_2$ *spans* AQ_n *except for the following cases: (a)* $l_1 = 2$ *with* $d_{AQ_n}(\mathbf{u},\mathbf{v}) = 1$ *such that* $\{\mathbf{x},\mathbf{y}\} = Nbd_{AQ_n}(\mathbf{u}) \cap Nbd_{AQ_n}(\mathbf{v})$; *(b)* $l_2 = 2$ *with* $d_{AQ_n}(\mathbf{x},\mathbf{y}) = 1$ *such that* $\{\mathbf{u},\mathbf{v}\} = Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$; *(c)* $l_1 = 2$ *with* $d_{AQ_n}(\mathbf{u},\mathbf{v}) = 2$ *such that* $\{\mathbf{x},\mathbf{y}\} = Nbd_{AQ_n}(\mathbf{u}) \cap Nbd_{AQ_n}(\mathbf{v})$; *and (d)* $l_2 = 2$ *with* $d_{AQ_n}(\mathbf{x},\mathbf{y}) = 2$ *such that* $\{\mathbf{u},\mathbf{v}\} = Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$.

Theorem 7. *Assume that* n *is a positive integer with* $n \geq 2$. *Then* AQ_n *satisfies 2RP-property.*

Proof. We prove this theorem by induction. By brute force, we check the theorem holds for $n = 2,3,4$. Assume the theorem holds for any AQ_k with $4 \leq k < n$. Without loss of generality, we can assume that $l_1 \geq l_2$. Thus, $l_2 \leq 2^{n-1} - 1$. By the symmetric property of AQ_n, we can assume that at least one of $\mathbf{u}$ and $\mathbf{v}$, say $\mathbf{u}$, is in $V(AQ^0_{n-1})$. Thus, we have the following cases:

Case 1: $\mathbf{v} \in V(AQ^0_{n-1})$ and $\{\mathbf{x},\mathbf{y}\} \subset V(AQ^1_{n-1})$.

Subcase 1.1: $d_{AQ_n}(\mathbf{x},\mathbf{y}) \leq l_2 \leq 2^{n-1} - 3$ except that (1) $l_2 = 2^{n-1} - 4$ and (2) $l_2 = 2$ if $d_{AQ_n}(\mathbf{x},\mathbf{y}) = 1$ or 2 with $\{\mathbf{u},\mathbf{v}\} \neq Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. See Figure 2(a) for an illustration. By Lemma 3, there exists a hamiltonian path R of AQ^0_{n-1} joining $\mathbf{u}$ to $\mathbf{v}$. Since $l(R) = 2^{n-1} - 1$, we can write R as $\langle \mathbf{u}, R_1, \mathbf{p}, \mathbf{q}, R_2, \mathbf{v} \rangle$ for some vertices $\mathbf{p}$ and $\mathbf{q}$ such that $\{\mathbf{p}^h, \mathbf{q}^h\} \cap \{\mathbf{x},\mathbf{y}\} = \emptyset$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}^h$ to $\mathbf{q}^h$ with $l(S_1) = 2^{n-1} - l_2 - 2$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(S_2) = l_2$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. We set

P_1 as $\langle \mathbf{u}, R_1, \mathbf{p}, \mathbf{p}^h, S_1, \mathbf{q}^h, \mathbf{q}, R_2, \mathbf{v} \rangle$ and set P_2 as S_2. Obviously, P_1 and P_2 are the required paths.

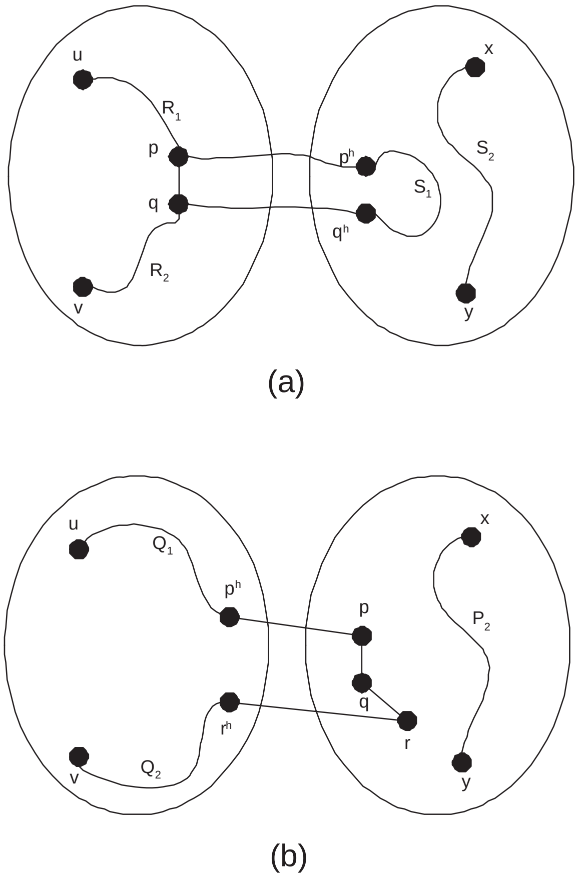

Figure 2: Subcase 1.1 and Subcase 1.3.

Subcase 1.2: $l_2 = 2$ if $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$ or 2 with $\{\mathbf{u}, \mathbf{v}\} \neq Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. Obviously, there exists a path P_2 of length 2 in $AQ_n - \{\mathbf{u}, \mathbf{v}\}$ joining $\mathbf{x}$ to $\mathbf{y}$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - V(P_2)$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths.

Subcase 1.3: $l_2 = 2^{n-1} - 4$. See Figure 2(b) for an illustration. Obviously, there exists a vertex $\mathbf{p}$ in $V(AQ^1_{n-1}) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}^h, \mathbf{v}^h\}$, a vertex $\mathbf{q}$ in $Nbd_{AQ^1_{n-1}}(\mathbf{p}) - \{\mathbf{x}, \mathbf{y}\}$, and a vertex $\mathbf{r}$ in $Nbd_{AQ^1_{n-1}}(\mathbf{q}) - \{\mathbf{x}, \mathbf{y}, \mathbf{p}\}$. Suppose that $\mathbf{r}^h \notin \{\mathbf{u}, \mathbf{v}\}$. By induction, there exist two disjoint paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{u}$ to $\mathbf{p}^h$, (2) Q_2 is a path joining $\mathbf{r}^h$ to $\mathbf{v}$, and (3) $Q_1 \cup Q_2$ spans AQ^0_{n-1}. By Lemma 3, there exists a hamiltonian path P_2 of $AQ^1_{n-1} - \{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ joining $\mathbf{x}$ to $\mathbf{y}$. We set P_1 as $\langle \mathbf{u}, Q_1, \mathbf{p}^h, \mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{r}^h, Q_2, \mathbf{v} \rangle$. Suppose that $\mathbf{r}^h \in \{\mathbf{u}, \mathbf{v}\}$. Without loss of generality, we assume that $\mathbf{r}^h = \mathbf{v}$. By Lemma 3, there exists a hamiltonian path R of $AQ^0_{n-1} - \{\mathbf{v}\}$ joining $\mathbf{u}$ to $\mathbf{p}^h$. We set P_1 as $\langle \mathbf{u}, R, \mathbf{p}^h, \mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{r}^h = \mathbf{v} \rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 1.4: $l_2 = 2^{n-1} - 2$. Obviously, there exist a vertex $\mathbf{p} \in V(AQ^1_{n-1}) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}^h, \mathbf{u}^c, \mathbf{v}^h, \mathbf{v}^c\}$. By Lemma 6, there exists two disjoint paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{u}$ and $\mathbf{p}^h$, (2) Q_2 is a path joining $\mathbf{p}^c$ and $\mathbf{v}$, and (3) $Q_1 \cup Q_2$ spans AQ^0_{n-1}. By Lemma 3, there exists a hamiltonian path P_2 of $AQ^0_{n-1} - \{\mathbf{p}\}$ joining $\mathbf{x}$ to $\mathbf{y}$. We set P_1 as $\langle \mathbf{u}, Q_1, \mathbf{p}^h, \mathbf{p}, \mathbf{p}^c, Q_2, \mathbf{v} \rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 1.5: $l_2 = 2^{n-1} - 1$. By Lemma 3, there exists a hamiltonian path P_1 of AQ^0_{n-1} joining $\mathbf{u}$ and $\mathbf{v}$ and there exists a hamiltonian path P_2 of AQ^1_{n-1} joining $\mathbf{x}$ to $\mathbf{y}$. Obviously, P_1 and P_2 are the required paths.

Case 2: $\mathbf{v} \in V(AQ^0_{n-1})$ and exactly one of $\mathbf{x}$ and $\mathbf{y}$ is in $V(AQ^0_{n-1})$. Without loss of generality, we assume that $\mathbf{x} \in V(AQ^0_{n-1})$.

Subcase 2.1: $l_2 = 1$. Obviously, $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$. We set P_2 as $\langle \mathbf{x}, \mathbf{y} \rangle$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - \{\mathbf{x}, \mathbf{y}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths.

Subcase 2.2: $l_2 = 2$ if $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$ or 2 with $\{\mathbf{u}, \mathbf{v}\} \neq Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. The proof is the same to Subcase 1.2.

Subcase 2.3: $l_2 = 3$.

Suppose that $d_{AQ_n}(\mathbf{x},\mathbf{y}) = 1$. There exists a vertex $\mathbf{p}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{x}) - \{\mathbf{u},\mathbf{v}\}$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - \{\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{p}^h\}$ joining $\mathbf{u}$ to $\mathbf{v}$. We set P_2 as $\langle \mathbf{x},\mathbf{p},\mathbf{p}^h,\mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Suppose that $d_{AQ_n}(\mathbf{x},\mathbf{y}) = 2$. By Lemma 4, there exists a path $\langle \mathbf{x},\mathbf{p},\mathbf{y}\rangle$ from $\mathbf{x}$ to $\mathbf{y}$ such that $\mathbf{p} \in V(AQ^1_{n-1})$. By Lemma 1, there exists a vertex $\mathbf{q} \in Nbd_{AQ^1_{n-1}}(\mathbf{p}) \cap Nbd_{AQ^1_{n-1}}(\mathbf{y})$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - \{\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. We set P_2 as $\langle \mathbf{x},\mathbf{p},\mathbf{q},\mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Suppose that $d_{AQ_n}(\mathbf{x},\mathbf{y}) = 3$. By Lemma 4, there exists a path P_2 from $\mathbf{x}$ to $\mathbf{y}$ such that $(V(P_2) - \{\mathbf{x}\}) \subset V(AQ^1_{n-1})$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - V(P_2)$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths.

Subcase 2.4: $4 \le l_2 \le 2^{n-1} - 2$ except that $l_2 = 2^{n-1} - 3$.

Suppose that $d_{AQ_n}(\mathbf{x},\mathbf{y}) = 1$ or 2. We first claim that there exists a vertex $\mathbf{p}$ in $Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. Assume that $d_{AQ_n}(\mathbf{x},\mathbf{y}) = 1$. Obviously, either $\mathbf{y} = \mathbf{x}^h$ or $\mathbf{y} = \mathbf{x}^c$. We set $\mathbf{p} = \mathbf{x}^c$ if $\mathbf{y} = \mathbf{x}^h$; and we set $\mathbf{p} = \mathbf{x}^h$ if $\mathbf{y} = \mathbf{x}^c$. Assume that $d_{AQ_n}(\mathbf{x},\mathbf{y}) = 2$. See Figure 3 for an illustration. By Lemma 4, there exists a path $\langle \mathbf{x},\mathbf{p},\mathbf{y}\rangle$ from $\mathbf{x}$ to $\mathbf{y}$ such that $\mathbf{p} \in V(AQ^1_{n-1})$. Obviously, $\mathbf{p}$ satisfies our claim. By Lemma 3, there exists a hamiltonian path R of $AQ^0_{n-1} - \{\mathbf{x}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. Since $l(R) = 2^{n-1} - 3$, we can write R as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{t}, R_2, \mathbf{v}\rangle$ such that $\{\mathbf{s}^h,\mathbf{t}^h\} \cap \{\mathbf{p},\mathbf{y}\} = \emptyset$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{s}^h$ to $\mathbf{t}^h$ with $l(S_1) = 2^{n-1} - 1 - l_2$, (2) S_2 is a path joining $\mathbf{p}$ to $\mathbf{y}$ with $l(S_2) = l_2 - 1$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^h, S_1, \mathbf{t}^h, \mathbf{t}, R_2, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, \mathbf{p}, S_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

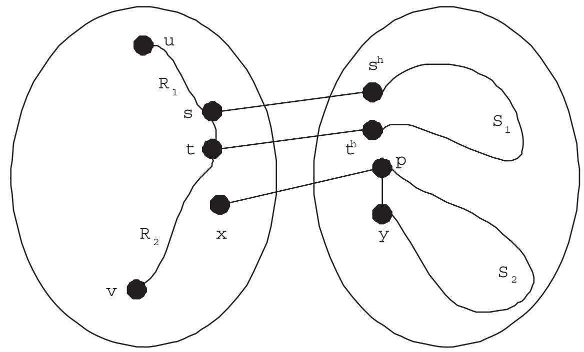

Figure 3: Subcase 2.4.

Suppose that $d_{AQ_n}(\mathbf{x},\mathbf{y}) \ge 3$. By Lemma 4, there exists a vertex $\mathbf{p}$ in $V(AQ^1_{n-1})$ such that $d_{AQ_n}(\mathbf{p},\mathbf{y}) = d_{AQ_n}(\mathbf{x},\mathbf{y}) - 1$. By Lemma 3, there exists a hamiltonian path R of $AQ^0_{n-1} - \{\mathbf{x}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. We can write R as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{t}, R_2, \mathbf{v}\rangle$ such that $\{\mathbf{s}^h,\mathbf{t}^h\} \cap \{\mathbf{p},\mathbf{y}\} = \emptyset$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{s}^h$ to $\mathbf{t}^h$ with $l(S_1) = 2^{n-1} - 1 - l_2$, (2) S_2 is a path joining $\mathbf{p}$ to $\mathbf{y}$ with $l(S_2) = l_2 - 1$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^h, S_1, \mathbf{t}^h, \mathbf{t}, R_2, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, \mathbf{p}, S_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 2.5: $l_2 = 2^{n-1} - 3$ or $l_2 = 2^{n-1} - 1$. Let $k = 3$ if $l_2 = 2^{n-1} - 3$ and $k = 1$ if $l_2 = 2^{n-1} - 1$. There exists a vertex $\mathbf{p}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{x}) - \{\mathbf{u},\mathbf{v},\mathbf{y}^n\}$. By Lemma 3, there exists a hamiltonian path R of $AQ^0_{n-1} - \{\mathbf{x},\mathbf{p}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. We can write R as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{t}, R_2, \mathbf{v}\rangle$ such that $\{\mathbf{s},\mathbf{t}\} \cap \{\mathbf{p},\mathbf{y}^n\} = \emptyset$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{s}^n$ to $\mathbf{t}^n$ with $l(S_1) = k$, (2) S_2 is a path joining $\mathbf{p}^n$ to $\mathbf{y}$ with $l(S_2) = 2^{n-1} - k - 2$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^n, S_1, \mathbf{t}^n, \mathbf{t}, R_2, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, \mathbf{p}, \mathbf{p}^n, S_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Case 3: $\{\mathbf{v}, \mathbf{x}, \mathbf{y}\} \subset V(Q_{n-1}^0)$.

Subcase 3.1: $l_2 = 1$. The proof is the same as Subcase 2.1.

Subcase 3.2: $l_2 = 2$ if $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$ or 2 with $\{\mathbf{u}, \mathbf{v}\} \neq Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. The proof is the same as Subcase 1.2.

Subcase 3.3: $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \leq l_2 \leq 2^{n-2} - 1$. See Figure 4(a) for an illustration. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(R_1) = 2^{n-1} - l_2 - 2$, (2) R_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(R_2) = l_2$, (3) $R_1 \cup R_2$ spans AQ_{n-1}^0. We can write R_1 as $\langle \mathbf{u}, R_3, \mathbf{p}, \mathbf{q}, R_4, \mathbf{v} \rangle$. By Lemma 3, there exists a hamiltonian path S of AQ_{n-1}^1 joining $\mathbf{p}^h$ to $\mathbf{q}^h$. We set P_1 as $\langle \mathbf{u}, R_3, \mathbf{p}, \mathbf{p}^h, S, \mathbf{q}^h, \mathbf{q}, R_4, \mathbf{v} \rangle$ and P_2 as R_2. Obviously, P_1 and P_2 are the required paths.

Subcase 3.4: $2^{n-2} + 1 \leq l_2 \leq 2^{n-1} - 1$ except that $l_2 = 2^{n-2} + 2$. See Figure 4(b) for an illustration. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(R_1) = 2^{n-2} - 1$, (2) R_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(R_2) = 2^{n-2} - 1$, and (3) $R_1 \cup R_2$ spans AQ_{n-1}^0. We can write R_1 as $\langle \mathbf{u}, R_3, \mathbf{p}, \mathbf{q}, R_4, \mathbf{v} \rangle$ and write R_2 as $\langle \mathbf{x}, R_5, \mathbf{s}, \mathbf{t}, R_6, \mathbf{y} \rangle$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}^h$ to $\mathbf{q}^h$ with $l(S_1) = 2^{n-1} - l_2 + 2^{n-2} - 2$, (2) S_2 is a path joining $\mathbf{s}^h$ to $\mathbf{t}^h$ with $l(S_2) = l_2 - 2^{n-2}$, and (3) $S_1 \cup S_2$ spans AQ_{n-1}^1. We set P_1 as $\langle \mathbf{u}, R_3, \mathbf{p}, \mathbf{p}^h, S_1, \mathbf{q}^h, \mathbf{q}, R_4, \mathbf{v} \rangle$ and P_2 as $\langle \mathbf{x}, R_5, \mathbf{s}, \mathbf{s}^h, S_2, \mathbf{t}^h, \mathbf{t}, R_6, \mathbf{y} \rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 3.5: $l_2 = 2^{n-2}$ or $2^{n-2} + 2$. Let $k = 0$ if $l_2 = 2^{n-2}$ and $k = 2$ if $l_2 = 2^{n-2} + 2$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(R_1) = 2^{n-2} - k$, (2) R_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(R_2) = 2^{n-2} + k - 2$, and (3) $R_1 \cup R_2$ spans AQ_{n-1}^0. We can write R_1 as $\langle \mathbf{u}, R_3, \mathbf{p}, \mathbf{q}, R_4, \mathbf{v} \rangle$ and write R_2 as $\langle \mathbf{x}, R_5, \mathbf{s}, \mathbf{t}, R_6, \mathbf{y} \rangle$. By Lemma 3, there exists a hamiltonian path S of $AQ_{n-1}^1 - \{\mathbf{s}^n, \mathbf{t}^n\}$ joining $\mathbf{p}^n$ to $\mathbf{q}^n$. We set P_1 as $\langle \mathbf{u}, R_3, \mathbf{p}, \mathbf{p}^n, S, \mathbf{q}^n, \mathbf{q}, R_4, \mathbf{v} \rangle$ and P_2 as $\langle \mathbf{x}, R_5, \mathbf{s}, \mathbf{s}^n, \mathbf{t}^n, \mathbf{t}, R_6, \mathbf{y} \rangle$. Obviously, P_1 and P_2 are the required paths.

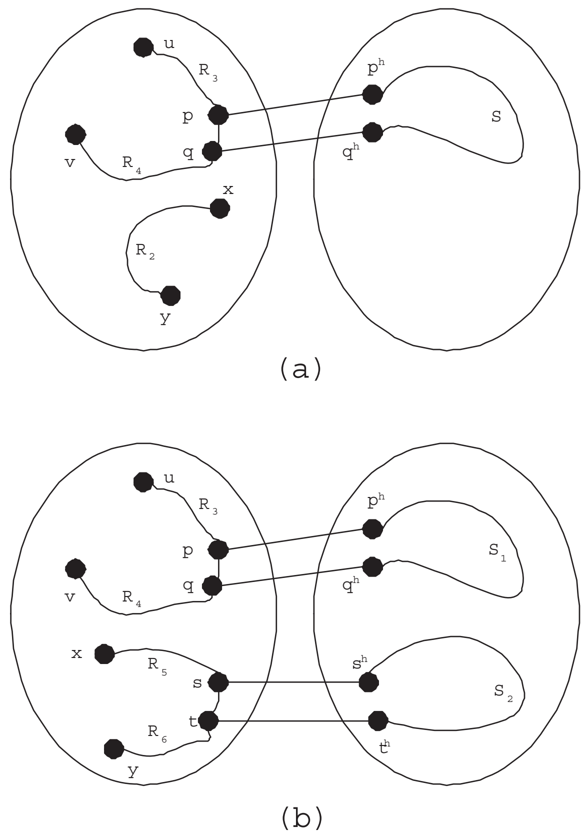

Figure 4: Subcase 3.3 and Subcase 3.4.

Case 4: $\{\mathbf{x}, \mathbf{v}, \mathbf{y}\} \subset V(AQ_{n-1}^1)$.

Subcase 4.1: $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \leq l_2 \leq 2^{n-1} - 3$ except that (1) $l_2 = 2^{n-1} - 4$ and (2) $l_2 = 2$ if $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$ or 2 with $\{\mathbf{u}, \mathbf{v}\} \neq Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. See Figure 5(a) for an illustration. Obviously, there exists a vertex $\mathbf{p}$ in $Nbd_{AQ_{n-1}^1}(\mathbf{v}) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}^h\}$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}$ to $\mathbf{v}$ with $l(S_1) = l_1 - 2^{n-1}$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(S_2) = l_2$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. By Lemma 3, there exists a hamiltonian path R of AQ^0_{n-1} joining $\mathbf{u}$ and $\mathbf{p}^h$. We set P_1 as $\langle \mathbf{u}, R, \mathbf{p}^h, \mathbf{p}, S_1, \mathbf{v}\rangle$ and we set P_2 as S_2. Obviously, P_1 and P_2 are the required paths.

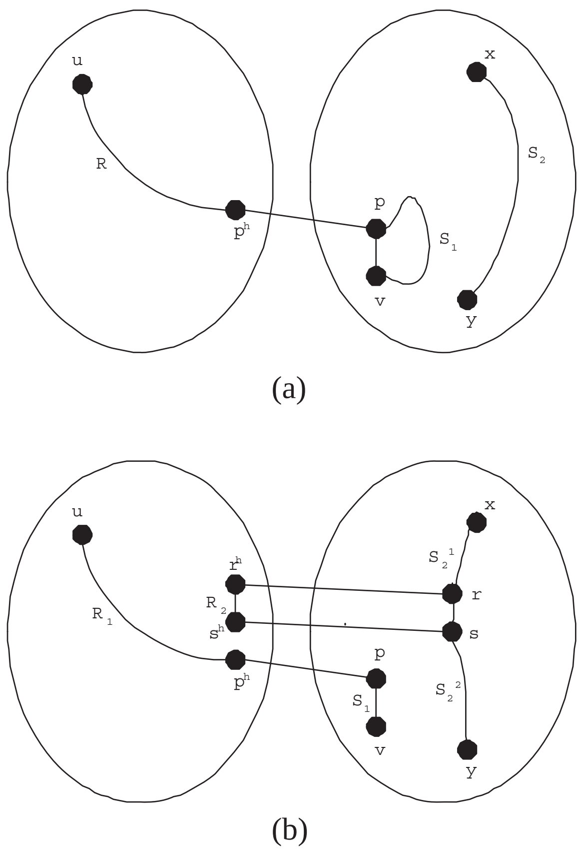

Figure 5: Subcase 4.1 and Subcase 4.5.

Subcase 4.2: $l_2 = 2$ if $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$ or 2 with $\{\mathbf{u}, \mathbf{v}\} \neq Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. The proof is the same to Subcase 1.2.

Subcase 4.3: $l_2 = 2^{n-1} - 4$. Obviously, there exists a vertex $\mathbf{p}$ in $Nbd_{AQ^1_{n-1}}(\mathbf{v}) - \{\mathbf{x}, \mathbf{y}\}$, and there exists a vertex $\mathbf{q}$ in $Nbd_{AQ^1_{n-1}}(\mathbf{p}) - \{\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{u}^h\}$. By Lemma 3, there exists a hamiltonian path R of AQ^0_{n-1} joining $\mathbf{u}$ to $\mathbf{q}^h$, and there exists a hamiltonian path P_2 of $AQ^1_{n-1} - \{\mathbf{v}, \mathbf{p}, \mathbf{q}\}$ joining $\mathbf{x}$ to $\mathbf{y}$. We set P_1 as $\langle \mathbf{u}, R, \mathbf{q}^h, \mathbf{q}, \mathbf{p}, \mathbf{v}\rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 4.4: $l_2 = 2^{n-1} - 2$. Let $\mathbf{v}'$ be an element in $\{\mathbf{v}^h, \mathbf{v}^c\} - \{\mathbf{u}\}$. By Lemma 3, there exists a hamiltonian path R of AQ^0_{n-1} joining $\mathbf{u}$ to $\mathbf{v}'$, and there exists a hamiltonian path P_2 of $AQ^1_{n-1} - \{\mathbf{v}\}$ joining $\mathbf{x}$ to $\mathbf{y}$. We set P_1 as $\langle \mathbf{u}, R, \mathbf{v}', \mathbf{v}\rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 4.5: $l_2 = 2^{n-1} - 1$. See Figure 5(b) for an illustration. Obviously, there exists a vertex $\mathbf{p}$ in $Nbd_{AQ^1_{n-1}}(\mathbf{v}) - \{\mathbf{x}, \mathbf{y}\}$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}$ to $\mathbf{v}$ with $l(S_1) = 1$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(S_2) = 2^{n-1} - 3$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. Obviously, we can write S_2 as $\langle \mathbf{x}, S^1_2, \mathbf{r}, \mathbf{s}, S^2_2, \mathbf{y}\rangle$ for some vertex $\mathbf{r}$ and $\mathbf{s}$ such that $\mathbf{u} \notin \{\mathbf{r}^h, \mathbf{s}^h\}$. Again by induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{p}^h$ with $l(R_1) = 2^{n-1} - 3$, (2) R_2 is a path joining $\mathbf{r}^h$ to $\mathbf{s}^h$ with $l(R_2) = 1$, and (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{p}^h, \mathbf{p}, \mathbf{v}\rangle$ and set P_2 as $\langle \mathbf{x}, S^1_2, \mathbf{r}, \mathbf{r}^h, \mathbf{s}^h, \mathbf{s}, S^2_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Case 5: $\mathbf{v} \in V(AQ^1_{n-1})$ and $|\{x, y\} \cap V(AQ^0_{n-1})| = 1$. Without loss of generality, we assume that $\mathbf{x} \in V(AQ^0_{n-1})$.

Subcase 5.1: $l_2 = 1$. The proof is the same to Subcase 2.1.

Subcase 5.2: $l_2 = 2$ if $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$ or 2 with $\{\mathbf{u}, \mathbf{v}\} \neq Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. The proof is the same to Subcase 1.2.

Subcase 5.3: $l_2 = 3$.

Suppose that $d_{AQ_n}(\mathbf{x},\mathbf{y}) = 1$. Obviously, there exists a vertex $\mathbf{p}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{x}) - \{\mathbf{u},\mathbf{v}^h\}$. We set P_2 as $\langle \mathbf{x},\mathbf{p},\mathbf{p}^h,\mathbf{y}\rangle$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - V(P_2)$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths.

Suppose that $d_{AQ_n}(\mathbf{x},\mathbf{y}) = 2$. Assume that $\{\mathbf{u},\mathbf{v}\} = Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. Thus, we have either $\mathbf{v} = \mathbf{x}^h$ or $\mathbf{v} = \mathbf{x}^c$. Moreover, $\mathbf{u} = \mathbf{x}^\alpha$, and $\mathbf{y} = \mathbf{v}^\alpha$ for some $\alpha \in \{i \mid 2 \le i \le n\} \cup \{i* \mid 2 \le i \le n-1\}$. We set P_2 as $\langle \mathbf{x},\mathbf{x}^{h*},(\mathbf{x}^{h*})^\alpha,((\mathbf{x}^h)^\alpha) = \mathbf{y}\rangle$ in the case of $\mathbf{v} = \mathbf{x}^h$. Otherwise, we set P_2 as $\langle \mathbf{x},\mathbf{x}^h,(\mathbf{x}^h)^\alpha,((\mathbf{x}^{h*})^\alpha) = \mathbf{y}\rangle$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - V(P_2)$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths. Now, assume that $\{\mathbf{u},\mathbf{v}\} \neq Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. By Lemma 1, there exists a vertex $\mathbf{p}$ in $(Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})) - \{\mathbf{u},\mathbf{v}\}$. Without loss of generality, we may assume that $\mathbf{p}$ is in AQ^0_{n-1}. By Lemma 1, there exists a vertex $\mathbf{q}$ in $(Nbd_{AQ^0_{n-1}}(\mathbf{p}) \cap Nbd_{AQ^0_{n-1}}(\mathbf{x})) - \{\mathbf{u}\}$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - \{\mathbf{x},\mathbf{q},\mathbf{p},\mathbf{y}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. We set P_2 as $\langle \mathbf{x},\mathbf{q},\mathbf{p},\mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Suppose that $d_{AQ_n}(\mathbf{x},\mathbf{y}) = 3$. By Lemma 4, there are two shortest paths R_1 and R_2 of AQ_n joining $\mathbf{x}$ to $\mathbf{y}$ such that R_1 can be written as $\langle \mathbf{x},\mathbf{r_1},\mathbf{r_2},\mathbf{y}\rangle$ with $\{\mathbf{r_1},\mathbf{r_2}\} \subset V(AQ^0_{n-1})$ and R_2 can be written as $\langle \mathbf{x},\mathbf{s_1},\mathbf{s_2},\mathbf{y}\rangle$ with $\{\mathbf{s_1},\mathbf{s_2}\} \subset V(AQ^1_{n-1})$. Suppose that $\mathbf{u} \neq \mathbf{r_2}$ or $\mathbf{v} \neq \mathbf{s_1}$. Without loss of generality, we assume that $\mathbf{u} \neq \mathbf{r_2}$. By Corollary 5, there exists a vertex $\mathbf{t} \in Nbd_{AQ^0_{n-1}}(\mathbf{x}) \cap Nbd_{AQ^0_{n-1}}(\mathbf{r_2}) - \{\mathbf{u}\}$. We set P_2 as $\langle \mathbf{x},\mathbf{t},\mathbf{r_2},\mathbf{y}\rangle$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - V(P_2)$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths. Thus, we consider $\mathbf{u} = \mathbf{r_2}$ and $\mathbf{v} = \mathbf{s_1}$. By Corollary 5, there exists a vertex $\mathbf{p}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{x}) \cap Nbd_{AQ^0_{n-1}}(\mathbf{u})$. Obviously, $d_{AQ_n}(\mathbf{p},\mathbf{y}) = 2$. By Lemma 4, there exists a vertex $\mathbf{q}$ in $V(AQ^1_{n-1}) \cap Nbd_{AQ_n}(\mathbf{p}) \cap Nbd_{AQ_n}(\mathbf{y})$. Since $d_{AQ_n}(\mathbf{q},\mathbf{y}) = 1$ and $d_{AQ_n}(\mathbf{v},\mathbf{y}) = 2$, $\mathbf{q} \neq \mathbf{v}$. We set P_2 as $\langle \mathbf{x},\mathbf{p},\mathbf{q},\mathbf{y}\rangle$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - V(P_2)$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths.

Subcase 5.4: $4 \le l_2 \le 2^{n-1} - 1$ with $d_{AQ_n}(\mathbf{x},\mathbf{y}) = 1$.

Suppose that $l_2 = 4$. See Figure 6(a) for an illustration. Obviously, there exists a vertex $\mathbf{p}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{x}) - \{\mathbf{u},\mathbf{v}^h\}$. By Lemma 1, there exists a vertex $\mathbf{q}$ in $(Nbd_{AQ^0_{n-1}}(\mathbf{x}) \cap Nbd_{AQ^0_{n-1}}(\mathbf{p})) - \{\mathbf{u}\}$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - \{\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{p}^h,\mathbf{q}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. We set P_2 as $\langle \mathbf{x},\mathbf{q},\mathbf{p},\mathbf{p}^h,\mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Suppose that $5 \le l_2 \le 2^{n-1} - 1$ except that $l_2 = 2^{n-1} - 2$. See Figure 6(b) for an illustration. Obviously, there exist a vertex $\mathbf{p}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{x}) - \{\mathbf{u},\mathbf{v}^h,\mathbf{y}^h\}$ and a vertex $\mathbf{s}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{u}) - \{\mathbf{x},\mathbf{p},\mathbf{v}^h,\mathbf{y}^h\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{s}$ with $l(R_1) = 2^{n-1} - 2 - l_2$, (2) R_2 is a path joining $\mathbf{p}$ to $\mathbf{x}$ with $l(R_2) = l_2 - 2$, and (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. By Lemma 3, there exists a hamiltonian path S of $AQ^1_{n-1} - \{\mathbf{y},\mathbf{p}^h\}$ joining $\mathbf{s}^h$ to $\mathbf{v}$. We set P_1 as $\langle \mathbf{u},R_1,\mathbf{s},\mathbf{s}^h,S,\mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x},R_2,\mathbf{p},\mathbf{p}^h,\mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Suppose that $l_2 = 2^{n-1} - 2$. See Figure 6(c) for an illustration. Let $\mathbf{s}$ and $\mathbf{p}$ be two vertices in $V(AQ^0_{n-1}) - \{\mathbf{u},\mathbf{x},\mathbf{v}^h,\mathbf{y}^h\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{s}$ with $l(R_1) = 2^{n-2}$, (2) R_2 is a path joining $\mathbf{p}$ to $\mathbf{x}$ with $l(R_2) = 2^{n-2} - 2$, (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. Similarly, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{s}^h$ to $\mathbf{v}$ with $l(S_1) = 2^{n-2} - 1$, (2) S_2 is a path joining $\mathbf{p}^h$ to $\mathbf{y}$ with $l(S_2) = 2^{n-2} - 1$, and (3)

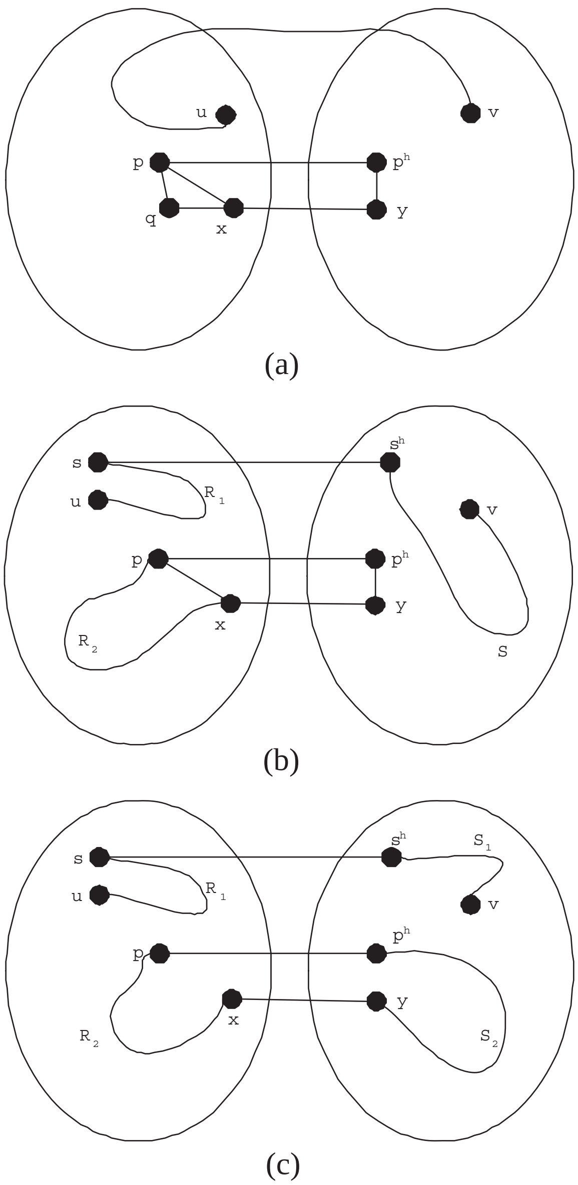

Figure 6: Subcase 5.4.

$S_1 \cup S_2$ spans AQ^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^h, S_1, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, R_2, \mathbf{p}, \mathbf{p}^h, S_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 5.5: $4 \le l_2 \le 2^{n-1} - 1$ except $l_2 = 2^{n-1} - 3$ with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \ge 2$.

Suppose that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 2$ with $\{\mathbf{u}, \mathbf{v}\} = Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. See Figure 7(a) for an illustration. Thus, we have either $\mathbf{v} = \mathbf{x}^h$ or $\mathbf{v} = \mathbf{x}^c$. Moreover, $\mathbf{u} = \mathbf{x}^\alpha$ and $\mathbf{y} = (\mathbf{x}^h)^\alpha$ for some $\alpha \in \{i \mid 2 \le i \le n\} \cup \{i* \mid 2 \le i \le n-1\}$. Obviously, there exists a vertex $\mathbf{t}$ in $Nbd_{AQ^1_{n-1}}(\mathbf{v}) - \{\mathbf{x}^h, \mathbf{y}, \mathbf{x}^c, \mathbf{u}^h\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{t}$ to $\mathbf{v}$ with $l(R_1) = 2^{n-1} - 1 - l_2$, (2) R_2 is a path joining $\mathbf{x}^c$ to $\mathbf{y}$ with $l(R_2) = l_2 - 1$ in the case of $\mathbf{v} = \mathbf{x}^h$; otherwise R_2 is a path joining $\mathbf{x}^h$ to $\mathbf{y}$ with $l(R_2) = l_2 - 1$, and (3) $R_1 \cup R_2$ spans AQ^1_{n-1}. By Lemma 3, there exists a hamiltonian path S of $AQ^0_{n-1} - \{\mathbf{x}\}$ joining $\mathbf{t^h}$ to $\mathbf{u}$. We set P_1 as $\langle \mathbf{u}, S, \mathbf{t}^h, \mathbf{t}, R_1, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, \mathbf{x}^c, R_2, \mathbf{y}\rangle$ in the case of $\mathbf{v} = \mathbf{x}^h$; otherwise, we set P_2 as $\langle \mathbf{x}, \mathbf{x}^h, R_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Suppose that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 2$ with $\{\mathbf{u}, \mathbf{v}\} \ne Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. See Figure 7(b) for an illustration. Then, there exists a vertex $\mathbf{p}$ in $(Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})) - \{\mathbf{u}, \mathbf{v}\}$. Without loss of generality, we may assume that $\mathbf{p} \in V(AQ^1_{n-1})$. Obviously, there exists a vertex $\mathbf{t}$ in $Nbd_{AQ^1_{n-1}}(\mathbf{v}) - \{\mathbf{y}, \mathbf{p}, \mathbf{u}^h, \mathbf{x}^h\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{t}$ to $\mathbf{v}$ with $l(R_1) = 2^{n-1} - 1 - l_2$, (2) R_2 is a path joining $\mathbf{p}$ to $\mathbf{y}$ with $l(R_2) = l_2 - 1$, and (3) $R_1 \cup R_2$ spans AQ^1_{n-1}. By Lemma 3, there exists a hamiltonian path S of $AQ^0_{n-1} - \{\mathbf{x}\}$ joining $\mathbf{t}^h$ to $\mathbf{u}$. We set P_1 as $\langle \mathbf{u}, S, \mathbf{t}^h, \mathbf{t}, R_1, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, \mathbf{p}, R_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Suppose that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = k \ge 3$. By

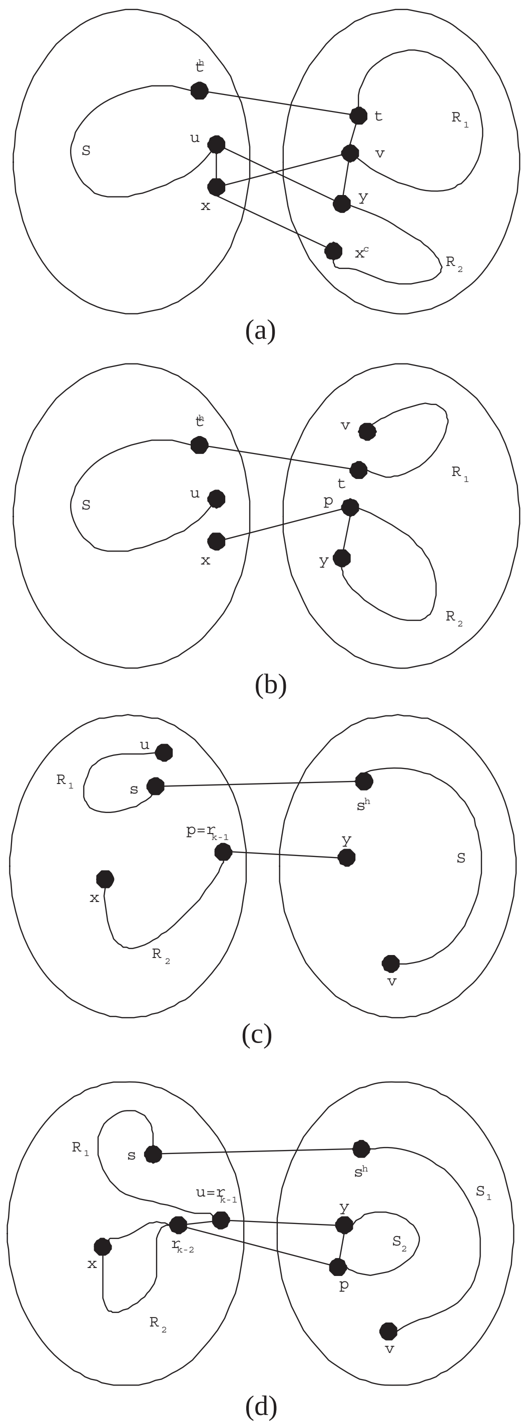

Figure 7: Subcase 5.5.

Lemma 4, there are two shortest paths S_1 and S_2 of AQ_n joining $\mathbf{x}$ to $\mathbf{y}$ such that S_1 can be written as $\langle \mathbf{x} = \mathbf{r_0}, \mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_{k-1}}, \mathbf{y} \rangle$ with $(V(S_1) - \{\mathbf{y}\}) \subset V(AQ^0_{n-1})$ and S_2 can be written as $\langle \mathbf{x}, \mathbf{s_1}, \mathbf{s_2}, \ldots, \mathbf{s_{k-1}}, \mathbf{y} \rangle$ with $(V(S_2) - \{\mathbf{x}\}) \subset V(AQ^1_{n-1})$. Suppose that $\mathbf{u} \neq \mathbf{r_{k-1}}$. See Figure 7(c) for an illustration. We set $\mathbf{p} = \mathbf{r_{k-1}}$. Again, there exists a vertex $\mathbf{s}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{u}) - \{\mathbf{x}, \mathbf{p}, \mathbf{y}^h, \mathbf{v}^h\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{s}$ with $l(R_1) = 2^{n-1} - 1 - l_2$, (2) R_2 is a path joining $\mathbf{p}$ to $\mathbf{x}$ with $l(R_2) = l_2 - 1$, and (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. By Lemma 3, there exists a hamiltonian path S of $AQ^1_{n-1} - \{\mathbf{y}\}$ joining $\mathbf{s}^h$ to $\mathbf{v}$. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^h, S, \mathbf{v} \rangle$ and P_2 as $\langle \mathbf{x}, R_2, \mathbf{p}, \mathbf{y} \rangle$. Obviously, P_1 and P_2 are the required paths.

Now we assume that $\mathbf{r_{k-1}} = \mathbf{u}$ and $\mathbf{s_1} = \mathbf{v}$. See Figure 7(d) for an illustration. Since $d_{AQ_n}(\mathbf{r_{k-2}}, \mathbf{y}) = 2$, by Lemma 4, there exists a vertex $\mathbf{p} \in Nbd_{AQ_n}(\mathbf{r_{k-2}})$ in $V(AQ^1_{n-1})$ such that $d_{AQ_n}(\mathbf{p}, \mathbf{y}) = 1$. Suppose that $l_2 = 4$ with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 3$. Thus, $\langle \mathbf{x}, \mathbf{r_1}, \mathbf{p}, \mathbf{y} \rangle$ is a shortest path joining $\mathbf{x}$ and $\mathbf{y}$. By Lemma 1, there exists a vertex $\mathbf{q} \in Nbd_{AQ^1_{n-1}}(\mathbf{p}) \cap Nbd_{AQ^1_{n-1}}(\mathbf{y}) - \{\mathbf{v}\}$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - \{\mathbf{x}, \mathbf{r_1}, \mathbf{p}, \mathbf{q}, \mathbf{y}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. We set P_2 as $\langle \mathbf{x}, \mathbf{r_1}, \mathbf{p}, \mathbf{q}, \mathbf{y} \rangle$. Obviously, P_1 and P_2 are the required paths. Suppose that $l_2 = 4$ with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 4$. Thus, $P_2 = \langle \mathbf{x}, \mathbf{r_1}, \mathbf{r_2}, \mathbf{p}, \mathbf{y} \rangle$ is a shortest path joining $\mathbf{x}$ and $\mathbf{y}$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - \{\mathbf{x}, \mathbf{r_1}, \mathbf{r_2}, \mathbf{p}, \mathbf{y}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths. Suppose that $5 \leq l_2 \leq 2^{n-2}$ with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \geq 3$. Obviously, there exists a vertex $\mathbf{s}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{u}) - \{\mathbf{x}, \mathbf{r_{k-2}}, \mathbf{y}^h, \mathbf{v}^h\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{s}$ with $l(R_1) = 2^{n-1} - l_2$, (2) R_2 is a path joining $\mathbf{r_{k-2}}$ to $\mathbf{x}$ with $l(R_2) = l_2 - 2$, and (3)

$R_1 \cup R_2$ spans AQ^0_{n-1}. By Lemma 3, there exists a hamiltonian path S of $AQ^1_{n-1} - \{\mathbf{p}, \mathbf{y}\}$ joining $\mathbf{s}^h$ to $\mathbf{v}$. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^h, S, \mathbf{v} \rangle$ and P_2 as $\langle \mathbf{x}, R_2, \mathbf{r_{k-2}}, \mathbf{p}, \mathbf{y} \rangle$. Obviously, P_1 and P_2 are the required paths. Suppose that $2^{n-2} + 1 \leq l_2 < 2^{n-1} - 1$ except $2^{n-1} - 3$ with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \geq 3$. Obviously, there exists a vertex $\mathbf{s}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{u}) - \{\mathbf{x}, \mathbf{r_{k-2}}, \mathbf{y}^h, \mathbf{v}^h\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{s}$ with $l(R_1) = 2^{n-2} + 1$, (2) R_2 is a path joining $\mathbf{r_{k-2}}$ to $\mathbf{x}$ with $l(R_2) = 2^{n-2} - 3$, and (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. Again by induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{s}^h$ to $\mathbf{v}$ with $l(S_1) = 2^{n-1} - l_2 + 2^{n-2} - 4$, (2) S_2 is a path joining $\mathbf{p}$ to $\mathbf{y}$ with $l(S_2) = l_2 - 2^{n-2} + 2$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^h, S_1, \mathbf{v} \rangle$ and P_2 as $\langle \mathbf{x}, R_2, \mathbf{r_{k-2}}, \mathbf{p}, S_2, \mathbf{y} \rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 5.6: $l_2 = 2^{n-1} - 3$ or $l_2 = 2^{n-1} - 1$ with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \geq 2$. Let $t = 0$ if $l_2 = 2^{n-1} - 3$ and $t = 1$ if $l_2 = 2^{n-1} - 1$. Obviously, there exist two vertices $\mathbf{s}$ and $\mathbf{p}$ in $AQ^0_{n-1} - \{\mathbf{u}, \mathbf{x}, \mathbf{v}^n, \mathbf{y}^n\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{s}$ with $l(R_1) = 2^{n-2} - t$, (2) R_2 is a path joining $\mathbf{p}$ to $\mathbf{x}$ with $l(R_2) = 2^{n-2} + t - 2$, and (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. Similarly, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{s}^n$ to $\mathbf{v}$ with $l(S_1) = 2^{n-2} - t$, (2) S_2 is a path joining $\mathbf{p}^n$ to $\mathbf{y}$ with $l(S_2) = 2^{n-2} + t - 2$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^n, S_1, \mathbf{v} \rangle$ and P_2 as $\langle \mathbf{x}, R_2, \mathbf{p}, \mathbf{p}^n, S_2, \mathbf{y} \rangle$. Obviously, P_1 and P_2 are the required paths.

Thus, Theorem 7 is proved. □

4 Concluding remarks

Now, we make some remarks to illustrate that some interesting properties of augmented cubes are consequences of Theorem 7.

Remark 1. The hamiltonian connected property of augmented cubes, proved in [8], states that there exists a hamiltonian path of AQ_n joining any two different vertices $\mathbf{u}$ and $\mathbf{y}$. Now, we prove that AQ_n is hamiltonian connected by Theorem 7. Obviously, AQ_n is hamiltonian connected for $n = 1$. Since $n \geq 2$, we can choose a pair of adjacent vertices $\mathbf{v}$ and $\mathbf{x}$ such that $\{\mathbf{v}, \mathbf{x}\} \cap \{\mathbf{u}, \mathbf{y}\} = \emptyset$. By Theorem 7, there are two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$, (2) P_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$, and (3) $P_1 \cup P_2$ spans AQ_n. Obviously, $\langle \mathbf{u}, P_1, \mathbf{v}, \mathbf{x}, P_2, \mathbf{y} \rangle$ forms a hamiltonian path joining $\mathbf{u}$ to $\mathbf{y}$. Thus, AQ_n is hamiltonian connected.

Remark 2. The panconnected property of AQ_n, proved in [13], stated that between any two different vertices $\mathbf{x}$ and $\mathbf{y}$ of AQ_n there exists a path $P_l(\mathbf{x}, \mathbf{y})$ of length l for any $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \leq l \leq 2^n - 1$. Now, we prove that AQ_n is panconnected by Theorem 7. Obviously, AQ_n is panconnected for $n = 1, 2$. Now, we consider that $n \geq 3$.

Suppose that $l = 2^n - 1$. By Remark 1, AQ_n is hamiltonian connected. Obviously, the hamiltonian path of AQ_n joining $\mathbf{x}$ and $\mathbf{y}$ is of length $2^n - 1$. Suppose that $l = 2^n - 2$. Let $\mathbf{u}$ be a vertex in $Nbd_{AQ_n}(\mathbf{y}) - \{\mathbf{x}\}$. By Lemma 1, there exists a vertex $\mathbf{v}$ in $(Nbd_{AQ_n}(\mathbf{u}) \cap Nbd_{AQ_n}(\mathbf{y})) - \{\mathbf{x}\}$. By Theorem 7, there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{x}$ to $\mathbf{u}$ with $l(P_1) = 2^n - 3$, (2) P_2 is a path joining $\mathbf{y}$ to $\mathbf{v}$ with $l(P_2) = 1$, and (3) $P_1 \cup P_2$ spans AQ_n. Obviously, $\langle \mathbf{x}, P_1, \mathbf{u}, \mathbf{y} \rangle$ is a path of length $2^n - 2$ joining $\mathbf{x}$ to $\mathbf{y}$. Suppose that $l = 2^n - 3$. We can find two adjacent vertices $\mathbf{u}$ and $\mathbf{v}$ such that $\{\mathbf{u}, \mathbf{v}\} \cap \{\mathbf{x}, \mathbf{y}\} = \emptyset$. By Theorem 7, there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(P_1) = 2^n - 3$, (2) P_2 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(P_2) = 1$, and (3)

$P_1 \cup P_2$ spans AQ_n. Obviously, P_1 is a path of length $2^n - 3$ joining $\mathbf{x}$ to $\mathbf{y}$. Suppose that $l \leq 2^n - 4$. By Lemma 2, there exist two vertices $\mathbf{u}$ and $\mathbf{v}$ such that $d_{AQ_n}(\mathbf{u}, \mathbf{v}) = 2$, $\{\mathbf{x}, \mathbf{y}\} \neq Nbd_{AQ_n}(\mathbf{u}) \cap Nbd_{AQ_n}(\mathbf{v})$, and $\{\mathbf{u}, \mathbf{v}\} \neq Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. By Theorem 7, there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(P_1) = l$, (2) P_2 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(P_2) = 2^n - 2 - l$, and (3) $P_1 \cup P_2$ spans AQ_n. Obviously, P_1 is a path of length l joining $\mathbf{x}$ to $\mathbf{y}$. Thus, AQ_n is panconnected.

Remark 3. The edge-pancyclic property property of AQ_n stated that for any edge $e = (\mathbf{x}, \mathbf{y})$ and for any $3 \leq l \leq 2^n$, there exists a cycle of length l containing e if $n \geq 2$. We prove that AQ_n is edge-pancyclic by Theorem 7. Obviously, AQ_n is edge-pancyclic for $n = 2$. Thus, we consider that $n \geq 3$.

Suppose that $l = 3$. By Lemma 1, there exists $\mathbf{u} \in Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. Obviously, $\langle \mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{x} \rangle$ forms a cycle of length three containing e. Now, we consider that $l = 2^n$ and $l = 2^n - 1$. By Lemma 1, there exists $\mathbf{v} \in (Nbd_{AQ_n}(\mathbf{u}) \cap Nbd_{AQ_n}(\mathbf{y})) - \{\mathbf{x}\}$. By Theorem 7, there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{x}$ to $\mathbf{u}$ with $l(P_1) = 2^n - 3$, (2) P_2 is a path joining $\mathbf{v}$ to $\mathbf{y}$ with $l(P_2) = 1$, and (3) $P_1 \cup P_2$ spans AQ_n. Obviously, $\langle \mathbf{x}, P_1, \mathbf{u}, \mathbf{v}, \mathbf{y}, \mathbf{x} \rangle$ forms a cycle of length 2^n containing e and $\langle \mathbf{x}, P_1, \mathbf{u}, \mathbf{y}, \mathbf{x} \rangle$ forms a cycle of length $2^n - 1$ containing e. Suppose $l = 2^n - 2$. By Theorem 7, there exist two disjoint paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(Q_1) = 2^n - 3$, (2) Q_2 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(Q_2) = 1$, and (3) $Q_1 \cup Q_2$ spans AQ_n. Obviously, $\langle \mathbf{x}, Q_1, \mathbf{y}, \mathbf{x} \rangle$ forms a cycle of length $2^n - 2$ containing e. Suppose that $4 \leq l \leq 2^n - 3$. By Lemma 2, there exists two vertices $\mathbf{p}$ and $\mathbf{q}$ of AQ_n such that $d_{AQ_n}(\mathbf{p}, \mathbf{q}) = 2$, $\{\mathbf{x}, \mathbf{y}\} \neq Nbd_{AQ_n}(\mathbf{p}) \cap Nbd_{AQ_n}(\mathbf{q})$, and $\{\mathbf{p}, \mathbf{q}\} \neq Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. By Theorem 7, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(R_1) = l - 1$, (2) R_2 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(R_2) = 2^n - l - 1$, and (3) $R_1 \cup R_2$ spans AQ_n. Obviously, $\langle \mathbf{x}, R_1, \mathbf{y}, \mathbf{x} \rangle$ forms a cycle of length l containing e.

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