# Cellular Networks Modeled by Distance Hereditary Graphs Are Maximum-Clique Perfect *

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***Abstract**-In modern cellular telecommunications systems, the entire service area of a country is divided into cells. Cells are normally thought of as hexagonal grids. One common method used to place transmitters for cellular telephones is to place them at the corner points of each hexagonal grid. Motivated by the placement of transmitters for cellular telephones, Chang, Kloks, and Lee introduced the concept of maximum-clique transversal sets on graphs in 2001. In this paper, we show that cellular networks modeled by distance-hereditary graphs are maximum-clique perfect. The maximum-clique transversal number and the maximum-clique independence number of a distance hereditary graph can be computed in linear time.*

**Keywords:** Algorithm, Maximum-Clique Transversal Set, Maximum-Clique Independent Set, Distance-Hereditary Graph..

## 1. Introduction

All graphs in this paper are undirected, finite, and simple. Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. For a graph $G$, we also use $V(G)$ and $E(G)$ to denote the vertex set and edge set of $G$, respectively. We use $G[W]$ to denote a subgraph of $G$ induced by a subset $W$ of $V$. For any vertex in $v \in V$, a *clique* is a subset of pairwise adjacent vertices of $V$. A *maximal clique* is a clique that is not a proper subset of any other clique. A clique is *maximum* if there is no clique of $G$ of larger cardinality. The *clique number* of $G$, denoted by $w(G)$, is the cardinality of a maximum clique of $G$. We use $Q(G)$ to denote the collection of all maximum cliques of $G$. A *maximum-clique transversal set* of a graph $G = (V, E)$ is a subset of $V$ intersecting all maximum cliques of $G$. The *maximum-clique transversal number* of $G$, denoted by $\tau_M(G)$, is the minimum cardinality of a maximum-clique transversal set of $G$. The *maximum-clique transversal set problem* is to find a maximum-clique transversal set of $G$ of minimum cardinality. A *maximum-clique independent set* of $G$ is a collection of pairwise disjoint maximum cliques of $G$. The *maximum-clique independence number* of $G$, denoted by $\alpha_M(G)$, is the maximum cardinality of a maximum-clique independent set of $G$. The *maximum-clique independent set problem* is to find a maximum-clique independent set of $G$ of maximum cardinality. It is clear that the *weak duality inequality* $\alpha_M(G) \le \tau_M(G)$ holds for any graph $G$.

*Clique transversal* and *clique independent* sets are closely related to maximum-clique transversal and maximum-clique independent sets. They have been studied in [3,4,5,6,7]. In this paper, we define a graph $G$ to be *maximum-clique perfect* if $\tau_M(H) = \alpha_M(H)$ for every induced subgraph $H$ of $G$.

A graph $G = (V,E)$ is called *distance hereditary* if every pair of vertices are equidistant in every connected induced subgraph containing them. The following theorem shows that distance-hereditary graphs can be defined recursively.

**Theorem 1.** [2] Distance-hereditary graphs can be defined recursively as follows:

1. A graph consisting of only one vertex is distance hereditary, and the twin set is the vertex itself.
2. If $G_1$ and $G_2$ are disjoint distance hereditary graphs with the twin sets $TS(G_1)$ and $TS(G_2)$, respectively, then the graph $G = G_1 \cup G_2$ is a distance-hereditary graph and the twin set of $G$ is $TS(G) = TS(G_1) \cup$

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$TS(G_2)$. $G$ is said to be obtained from $G_1$ and $G_2$ by a *false twin* operation.

3. If $G_1$ and $G_2$ are disjoint distance hereditary graphs with the twin sets $TS(G_1)$ and $TS(G_2)$, respectively, then the graph $G$ obtained by connecting every vertex of $TS(G_1)$ to all vertices of $TS(G_2)$ is a distance hereditary graph, and the twin set of $G$ is $TS(G) = TS(G_1) \cup TS(G_2)$. $G$ is said to be obtained from $G_1$ and $G_2$ by a *true twin* operation.
4. If $G_1$ and $G_2$ are disjoint distance hereditary graphs with the twin sets $TS(G_1)$ and $TS(G_2)$, respectively, then the graph $G$ obtained by connecting every vertex of $TS(G_1)$ to all vertices of $TS(G_2)$ is a distance hereditary graph, and the twin set of $G$ is $TS(G) = TS(G_1)$. $G$ is said to be obtained from $G_1$ and $G_2$ by a *pendant vertex* operation.

Following **Theorem 1**, a binary ordered decomposition tree can be obtained in linear time [1]. In this decomposition tree, each leaf is a single vertex graph, and each internal node represents one of the three operations: pendant vertex operation (labeled by P), true twin operation (labeled by T), and false twin operation (labeled by F). This ordered decomposition tree is called a **PTF**-tree.

## 2. Main Result

In this section, we will prove that distance hereditary graphs are maximum-clique perfect. Due to space limitations, we have to omit the proof of each lemma and theorem in this section.

**Definition 1**. Recall that $Q(G)$ denotes the collection of all maximum cliques of $G$. Hence $Q(G[TS(G)])$ is the collection of all maximum cliques of $G[TS(G)]$. We use $Q_{TS}(G)$ to denote the collection of all maximum cliques of $G$ which are maximum cliques of $G[TS(G)]$ and use $Q_{\overline{TS}}(G)$ to denote the collection of all maximum cliques of $G$ which are not maximum cliques of $G[TS(G)]$. Hence $Q(G) = Q_{TS}(G) \cup Q_{\overline{TS}}(G)$. Let $Q_E(G) = Q(G) \cup Q(G[TS(G)])$. $Q_E(G)$ denotes the collection of all maximum cliques of $G$ and all maximum cliques of $G[TS(G)]$.

**Definition 2.** Suppose that a distance hereditary graph $G$ is obtained from two disjoint distance hereditary graphs $G_1$ and $G_2$ by one of the three operations: pendant vertex operation, true twin operation, and false twin operation. We use $w$, $w_t$, $w_1$, $w_{t_1}$, $w_2$, and $w_{t_2}$ to denote the clique numbers of $G$, $G[TS(G)]$, $G_1$, $G_1[TS(G_1)]$, $G_2$, and $G_2[TS(G_2)]$, respectively.

**Lemma 1.** Suppose that $G$ is a graph obtained from two disjoint distance hereditary graphs $G_1$ and $G_2$ by a false twin operation. Let $i \in \{1,2\}$. Then,

(1) $$Q(G[TS(G)] = \begin{cases} Q(G_1[TS(G_1)]) \text{ if } w_{t_1} > w_{t_2}; \\ Q(G_2[TS(G_2)]) \text{ if } w_{t_2} > w_{t_1}; \\ Q(G_1[TS(G_1)]) \cup Q(G_2[TS(G_2)]) \\ \text{if } w_{t_1} = w_{t_2}. \end{cases}$$

(2) $$Q(G) = \begin{cases} Q(G_1) \text{ if } w_1 > w_2; \\ Q(G_2) \text{ if } w_2 > w_1; \\ Q(G_1) \cup Q(G_2) \text{ if } w_1 = w_2. \end{cases}$$

(3) $$Q_{TS}(G) = \begin{cases} Q_{TS}(G_1) \text{ if } w_1 > w_2; \\ Q_{TS}(G_2) \text{ if } w_2 > w_1; \\ Q_{TS}(G_1) \cup Q_{TS}(G_2) \text{ if } w_1 = w_2. \end{cases}$$

(4) $$Q_{\overline{TS}}(G) = \begin{cases} Q_{\overline{TS}}(G_1) \text{ if } w_1 > w_2; \\ Q_{\overline{TS}}(G_2) \text{ if } w_2 > w_1; \\ Q_{\overline{TS}}(G_1) \cup Q_{\overline{TS}}(G_2) \text{ if } w_1 = w_2. \end{cases}$$

(5) $$Q_E(G) = \begin{cases} Q_E(G_1) \cup Q_E(G_2) \\ \text{if } w_{t_1} = w_{t_2} \text{ and } w_1 = w_2; \\ Q_E(G_i) \cup Q(G_{3-i}[TS(G_{3-i})]) \\ \text{if } w_{t_1} = w_{t_2} \text{ and } w_i < w_{3-i}; \\ Q_E(G_i) \cup Q_{\overline{TS}}(G_{3-i}) \\ \text{if } w_{t_i} > w_{t_{3-i}} \text{ and } w_1 = w_2; \\ Q(G_i[TS(G_i)]) \cup Q_{\overline{TS}}(G_{3-i}) \\ \text{if } w_{t_i} > w_{t_{3-i}} \text{ and } w_i < w_{3-i}; \\ Q_E(G_i) \text{ if } w_{t_i} > w_{t_{3-i}} \text{ and } w_i > w_{3-i}; \end{cases}$$

**Definition 3**. Suppose that $G$ is a graph obtained from two disjoint distance hereditary graphs $G_1$ and $G_2$ by a true twin operation or a pendant vertex operation. We use $Q_{12}(G)$ to denote $\{q_1 \cup q_2 \mid q_1 \in Q(G_1[TS(G_1)]) \text{ and } q_2 \in Q(G_2[TS(G_2)])\}$.

**Lemma 2.** Suppose that $G$ is a graph obtained from two disjoint distance hereditary graph $G_1$ and $G_2$ by a true twin operation. Then,

(1) $Q(G[TS(G)]) = Q_{12}(G)$.

(2)

$$Q(G)=\begin{cases} Q_{12}(G) \text{ if } w_{t_1}+w_{t_2}>\max\{w_1,w_2\}; \\ Q_{12}(G)\cup Q_{\overline{TS}}(G_1) \text{ if } w_{t_1}+w_{t_2}=w_1>w_2; \\ Q_{12}(G)\cup Q_{\overline{TS}}(G_2) \text{ if } w_{t_1}+w_{t_2}=w_2>w_1; \\ Q_{12}(G)\cup Q_{\overline{TS}}(G_1)\cup Q_{\overline{TS}}(G_2) \\ \text{if } w_{t_1}+w_{t_2}=w_1=w_2; \\ Q(G_1)=Q_{\overline{TS}}(G_1) \\ \text{if } w_{t_1}+w_{t_2}<w_1 \text{ and } w_1>w_2; \\ Q(G_2)=Q_{\overline{TS}}(G_2) \\ \text{if } w_{t_1}+w_{t_2}<w_2 \text{ and } w_2>w_1; \\ Q_{\overline{TS}}(G_1)\cup Q_{\overline{TS}}(G_2) \\ \text{if } w_{t_1}+w_{t_2}<w_1=w_2. \end{cases}$$

(3)

$$Q_{\overline{TS}}(G)=\begin{cases} \phi \text{ if } w_{t_1}+w_{t_2}>\max\{w_1,w_2\}; \\ Q_{\overline{TS}}(G_1) \text{ if } w_{t_1}+w_{t_2}=w_1>w_2; \\ Q_{\overline{TS}}(G_2) \text{ if } w_{t_1}+w_{t_2}=w_2>w_1; \\ Q_{\overline{TS}}(G_1)\cup Q_{\overline{TS}}(G_2) \\ \text{if } w_{t_1}+w_{t_2}=w_1=w_2; \\ Q(G_1)=Q_{\overline{TS}}(G_1) \\ \text{if } w_{t_1}+w_{t_2}<w_1 \text{ and } w_1>w_2; \\ Q(G_2)=Q_{\overline{TS}}(G_2) \\ \text{if } w_{t_1}+w_{t_2}<w_2 \text{ and } w_2>w_1; \\ Q_{\overline{TS}}(G_1)\cup Q_{\overline{TS}}(G_2) \\ \text{if } w_{t_1}+w_{t_2}<w_1=w_2. \end{cases}$$

(4)

$$Q_{TS}(G)=\begin{cases} Q(G[TS(G)]) \text{ if } w_{t_1}+w_{t_2}\geq\max\{w_1,w_2\}; \\ \phi, \text{otherwise.} \end{cases}$$

(5) $$Q_E(G)=\begin{cases} Q(G) \text{ if } w_{t_1}+w_{t_2}\geq\max\{w_1,w_2\}; \\ Q(G[TS(G)])\cup Q(G), \text{otherwise.} \end{cases}$$

**Lemma 3.** Suppose that $G$ is a graph obtained from two disjoint distance hereditary graph $G_1$ and $G_2$ by a true twin operation. Then,

(1)

$$Q(G)=\begin{cases} Q_{12}(G) \text{ if } w_{t_1}+w_{t_2}>\max\{w_1,w_2\}; \\ Q_{12}(G)\cup Q_{\overline{TS}}(G_1) \text{ if } w_{t_1}+w_{t_2}=w_1>w_2; \\ Q_{12}(G)\cup Q_{\overline{TS}}(G_2) \\ \text{if } w_{t_1}+w_{t_2}=w_2>w_1; \\ Q_{12}(G)\cup Q_{\overline{TS}}(G_1)\cup Q_{\overline{TS}}(G_2) \\ \text{if } w_{t_1}+w_{t_2}=w_1=w_2; \\ Q(G_1)=Q_{\overline{TS}}(G_1) \\ \text{if } w_{t_1}+w_{t_2}<w_1 \text{ and } w_1>w_2; \\ Q(G_2)=Q_{\overline{TS}}(G_2) \\ \text{if } w_{t_1}+w_{t_2}<w_2 \text{ and } w_2>w_1; \\ Q_{\overline{TS}}(G_1)\cup Q_{\overline{TS}}(G_2) \\ \text{if } w_{t_1}+w_{t_2}<w_1=w_2. \end{cases}$$

(2)

$$Q_E(G)=\begin{cases} Q_{12}(G)\cup Q(G_1[TS(G_1)]) \\ \text{ if } w_{t_1}+w_{t_2}>\max\{w_1,w_2\}; \\ Q_{12}(G)\cup Q_E(G_1) \text{ if } w_{t_1}+w_{t_2}=w_1>w_2; \\ Q(G_1[TS(G_1)])\cup Q_{12}(G)\cup Q_{\overline{TS}}(G_2) \\ \text{if } w_{t_1}+w_{t_2}=w_2>w_1; \\ Q_{12}(G)\cup Q_E(G_1)\cup Q_{\overline{TS}}(G_2) \\ \text{if } w_{t_1}+w_{t_2}=w_1=w_2; \\ Q_E(G_1) \\ \text{if } w_{t_1}+w_{t_2}<w_1 \text{ and } w_1>w_2; \\ Q(G_1[TS(G_1)])\cup Q_{\overline{TS}}(G_2) \\ \text{if } w_{t_1}+w_{t_2}<w_2 \text{ and } w_2>w_1; \\ Q_E(G_1)\cup Q_{\overline{TS}}(G_2) \\ \text{if } w_{t_1}+w_{t_2}<w_1=w_2. \end{cases}$$

(3) $Q(G[TS(G)])=Q(G_1[TS(G_1)])$ , $Q_{TS}(G)=\phi$ , and $Q_{\overline{TS}}(G)=Q(G)$.

**Lemma 4.** Suppose that $G$ is a graph obtained from two disjoint distance hereditary graphs $G_1$ and $G_2$ by a true twin operation or a pendant vertex operation. Let $S$ be a maximum-clique transversal set of $G$ . If $w_{t_1}+w_{t_2}\geq\max\{w_1,w_2\}$, then either $S\cap TS(G_1)$ is a maximum-clique transversal set of $G_1[TS(G_1)]$ or $S\cap TS(G_2)$ is a maximum-clique transversal set of $G_2[TS(G_2)]$.

**Definition 4.** A *strong maximum-clique transversal set* of $G$ is a subset of $V$ that intersects all cliques in $Q_E(G)$ . We use $SCT(G)$ to represent a strong maximum-clique transversal set of $G$.

**Definition 5.** A *weak maximum-clique transversal set* of $G$ is a subset of $V$ that intersects all cliques in $Q_{\overline{TS}}(G)$. We use $WCT(G)$ to represent a weak maximum-clique transversal set of $G$.

**Definition 6.** An *expanded maximum-clique independent set* of $G$ is a collection of pairwise disjoint cliques in $Q_E(G)$. We use $ECI(G)$ to represent an expanded maximum-clique independent set of $G$.

**Definition 7.** A *weak maximum-clique independent set* of $G$ is a collection of pairwise disjoint cliques in $Q_{\overline{TS}}(G)$. We use $WCI(G)$ to represent a weak maximum-clique independent set of $G$.

**Definition 8.** Let $CT(G)$ and $CI(G)$ denote a maximum-clique transversal set and a maximum-clique independent set of $G$, respectively. We say that a distance hereditary graph $G$ holds the *strong duality* if there exist a $CT(G)$, a $CI(G)$, a $CT(G[TS(G)])$, a $CI(G[TS(G)])$, a $WCT(G)$, a $WCI(G)$, an $SCT(G)$, and an $ECI(G)$ such that the following five conditions are satisfied: (1) $|CT(G)| = |CI(G)|$, (2) $|CT(G[TS(G)])| = |CI(G[TS(G)])|$, (3) $|WCT(G)| = |WCI(G)|$, (4) $|SCT(G)| = |ECI(G)|$, and (5) $WCI(G)$ is a subset of $ECI(G)$. Let $XI(G)$ denote $ECI(G)-WCI(G)$.

**Definition 9.** Assume that $G$ is a distance hereditary graph formed from two disjoint distance hereditary graphs $G_1$ and $G_2$ by a pendant vertex operation or by a true twin operation, and both $G_1$ and $G_2$ hold the strong duality. Suppose that $XI(G_1) = \{c_1,\ldots,c_{k_1}\}$, $XI(G_2) = \{d_1,\ldots,d_{k_2}\}$, $CI(G_1[TS(G_1)]) = \{p_1,\ldots p_{r_1}\}$, and $CI(G_1[TS(G_1)]) = \{q_1,\ldots q_{r_2}\}$. We have the following definitions.

(1) Let $k = \min\{k_1, k_2\}$. We let $XX(G) = \{c_i \cup d_i \mid 1 \le i \le k\}$ and let $XX'(G) = \{c_{k+1},\ldots,c_{k_1}\}$ if $k_1 > k$ and $XX'(G) = \phi$ otherwise.

(2) Let $r = \min\{r_1, r_2\}$. We let $TT(G) = \{p_i \cup q_i \mid 1 \le i \le r\}$ and let $TT'(G) = \{p_{r+1},\ldots,p_{r_1}\}$ if $r_1 > r$ and $TT'(G) = \phi$ otherwise.

(3) Let $\ell = \min\{k_1, r_2\}$. We let $XT(G) = \{c_i \cup q_i \mid 1 \le i \le \ell\}$ and let $XT'(G) = \{c_{\ell+1},\ldots,c_{k_1}\}$ if $k_1 > \ell$ and $XT'(G) = \phi$ otherwise.

(4) Let $s = \min\{r_1, k_2\}$. We let $TX(G) = \{p_i \cup d_i \mid 1 \le i \le s\}$ and let $TX'(G) = \{p_{s+1},\ldots,p_{r_1}\}$ if $r_1 > s$ and $TX'(G) = \phi$ otherwise.

**Lemma 5.** Assume that $G$ is a graph of single vertex and $v$ is the vertex of $G$. There exist the following sets: (1) $CT(G)=\{v\}$, (2) $CT(G[TS(G)])=\{v\}$, (3) $SCT(G)=\{v\}$, (4) $WCT(G)=\phi$, (5) $WCI(G) = \phi$, (6) $ECI(G) = \{v\}\}$, (7) $CI(G[TS(G)]) = \{\{v\}\}$, and (8) $CI(G) = \{\{v\}\}$ such that $G$ holds the strong duality.

**Lemma 6.** Assume that $G$ is obtained from two disjoint distance hereditary graphs $G_1$ and $G_2$ by a false twin operation , and both $G_1$ and $G_2$ hold the strong duality. Let $i \in \{1,2\}$. There exist the following sets such that $G$ holds the strong duality.

(1)
$$CT(G)=\begin{cases} CT(G_1) \text{ if } w_1 > w_2; \\ CT(G_2) \text{ if } w_2 > w_1; \\ CT(G_1)\cup CT(G_2) \text{ if } w_1 = w_2. \end{cases}$$

(2)
$$CT(G[TS(G)])=\begin{cases} CT(G_1[TS(G_1)]) \text{ if } w_{t_1} > w_{t_2}; \\ CT(G_2[TS(G_2)]) \text{ if } w_{t_2} > w_{t_1}; \\ CT(G_1[TS(G_1)])\cup CT(G_2[TS(G_2)]) \\ \text{ if } w_{t_2} = w_{t_1}. \end{cases}$$

(3)
$$SCT(G)=\begin{cases} SCT(G_1)\cup SCT(G_2) \\ \text{if } w_{t_1} = w_{t_2} \text{ and } w_1 = w_2; \\ SCT(G_i)\cup CT(G_{3-i}[TS(G_{3-i})]) \\ \text{if } w_{t_1} = w_{t_2} \text{ and } w_i > w_{3-i}; \\ SCT(G_i)\cup WCT(G_{3-i}) \\ \text{if } w_{t_i} > w_{t_{3-i}} \text{ and } w_1 = w_2; \\ CT(G_i[TS(G_i)])\cup WCT(G_{3-i}) \\ \text{if } w_{t_i} > w_{t_{3-i}} \text{ and } w_i < w_{3-i}; \\ SCT(G_i) \\ \text{if } w_{t_i} > w_{t_{3-i}} \text{ and } w_i > w_{3-i}; \end{cases}$$

(4)
$$WCT(G)=\begin{cases} WCT(G_1) \text{ if } w_1 > w_2; \\ WCT(G_2) \text{ if } w_2 > w_1; \\ WCT(G_1)\cup WCT(G_2) \text{ if } w_1 = w_2. \end{cases}$$

(5)
$$WCI(G)=\begin{cases} WCI(G_1) \text{ if } w_1 > w_2; \\ WCI(G_2) \text{ if } w_2 > w_1; \\ WCI(G_1)\cup WCI(G_2) \text{ if } w_1 = w_2. \end{cases}$$

(6)
$$ECI(G)=\begin{cases} ECI(G_1)\cup ECI(G_2) \\ \text{if } w_{t_1} = w_{t_2} \text{ and } w_1 = w_2; \\ ECI(G_i)\cup CI(G_{3-i}[TS(G_{3-i})]) \\ \text{if } w_{t_1} = w_{t_2} \text{ and } w_i > w_{3-i}; \\ ECI(G_i)\cup WCI(G_{3-i}) \\ \text{if } w_{t_i} > w_{t_{3-i}} \text{ and } w_1 = w_2; \\ CI(G_i[TS(G_i)])\cup WCI(G_{3-i}) \\ \text{if } w_{t_i} > w_{t_{3-i}} \text{ and } w_i < w_{3-i}; \\ ECI(G_i) \\ \text{if } w_{t_i} > w_{t_{3-i}} \text{ and } w_i > w_{3-i}; \end{cases}$$

(7)
$$CI(G[TS(G)])=\begin{cases} CI(G_1[TS(G_1)]) \text{ if } w_{t_1} > w_{t_2}; \\ CI(G_2[TS(G_2)]) \text{ if } w_{t_2} > w_{t_1}; \\ CI(G_1[TS(G_1)])\cup CI(G_2[TS(G_2)]) \\ \text{ if } w_{t_2} = w_{t_1}. \end{cases}$$

(8)
$$CI(G)=\begin{cases} CI(G_1) \text{ if } w_1 > w_2; \\ CI(G_2) \text{ if } w_2 > w_1; \\ CI(G_1)\cup CI(G_2) \text{ if } w_1 = w_2. \end{cases}$$

**Lemma 7.** Assume that $G$ is obtained from two disjoint distance hereditary graphs $G_1$ and $G_2$ by a pendant vertex operation , and both $G_1$ and $G_2$ hold the strong duality.

$$(1)\ CT(G)=\begin{cases}\min\{CT(G_1[TS(G_1)]),CT(G_2[TS(G_2)])\}\\ \text{ if } w_{t_1}+w_{t_2}>\max\{w_1,w_2\};\\ \min\{SCT(G_1),WCT(G_1)\cup CT(G_2[TS(G_2)])\}\\ \text{if } w_{t_1}+w_{t_2}=w_1>w_2;\\ \min\{SCT(G_2),WCT(G_2)\cup CT(G_1[TS(G_1)])\}\\ \text{if } w_{t_1}+w_{t_2}=w_2>w_1;\\ \min\{SCT(G_1)\cup WCT(G_2),\\ WCT(G_1)\cup SCT(G_2)\}\\ \text{if } w_{t_1}+w_{t_2}=w_1=w_2;\\ CT(G_1)=WCT(G_1)\\ \text{if } w_{t_1}+w_{t_2}<w_1 \text{ and } w_1>w_2;\\ CT(G_2)=WCT(G_2)\\ \text{if } w_{t_1}+w_{t_2}<w_2 \text{ and } w_2>w_1;\\ WCT(G_1)\cup WCT(G_2)\\ \text{if } w_{t_1}+w_{t_2}<w_1=w_2.\end{cases}$$

(2) $CT(G[TS(G)])=CT(G_1[TS(G_1)])$.

(3) $WCT(G) = CT(G)$.

$$(4)\ SCT(G)=\begin{cases}CT(G_1[TS(G_1)])\\ \text{ if } w_{t_1}+w_{t_2}>\max\{w_1,w_2\};\\ SCT(G_1)\\ \text{if } w_{t_1}+w_{t_2}=w_1>w_2;\\ WCT(G_2)\cup CT(G_1[TS(G_1)])\\ \text{if } w_{t_1}+w_{t_2}=w_2>w_1;\\ SCT(G_1)\cup WCT(G_2)\\ \text{if } w_{t_1}+w_{t_2}=w_1=w_2;\\ SCT(G_1)\\ \text{if } w_{t_1}+w_{t_2}<w_1 \text{ and } w_1>w_2;\\ WCT(G_2)\cup CT(G_1[TS(G_1)])\\ \text{if } w_{t_1}+w_{t_2}<w_2 \text{ and } w_2>w_1;\\ SCT(G_1)\cup WCT(G_2)\\ \text{if } w_{t_1}+w_{t_2}<w_1=w_2.\end{cases}$$

$$(5)\ WCI(G)=\begin{cases}TT(G)\\ \text{ if } w_{t_1}+w_{t_2}>\max\{w_1,w_2\};\\ WCI(G_1)\cup XT(G)\\ \text{if } w_{t_1}+w_{t_2}=w_1>w_2;\\ WCI(G_2)\cup TX(G)\\ \text{if } w_{t_1}+w_{t_2}=w_2>w_1;\\ WCI(G_1)\cup WCI(G_2)\cup XX(G)\\ \text{if } w_{t_1}+w_{t_2}=w_1=w_2;\\ WCI(G_1)\\ \text{if } w_{t_1}+w_{t_2}<w_1 \text{ and } w_1>w_2;\\ WCI(G_2)\\ \text{if } w_{t_1}+w_{t_2}<w_2 \text{ and } w_2>w_1;\\ WCI(G_1)\cup WCI(G_2)\\ \text{if } w_{t_1}+w_{t_2}<w_1=w_2.\end{cases}$$

$$(6)\ ECI(G)=\begin{cases}TT(G)\cup TT'(G)\\ \text{ if } w_{t_1}+w_{t_2}>\max\{w_1,w_2\};\\ WCI(G_1)\cup XT(G)\cup XT'(G)\\ \text{if } w_{t_1}+w_{t_2}=w_1>w_2;\\ WCI(G_2)\cup TX(G)\cup TX'(G)\\ \text{if } w_{t_1}+w_{t_2}=w_2>w_1;\\ WCI(G_1)\cup WCI(G_2)\cup\\ XX(G)\cup XX'(G)\\ \text{if } w_{t_1}+w_{t_2}=w_1=w_2;\\ ECI(G_1)\\ \text{if } w_{t_1}+w_{t_2}<w_1 \text{ and } w_1>w_2;\\ WCI(G_2)\cup CI(G_1[TS(G_1)])\\ \text{if } w_{t_1}+w_{t_2}<w_2 \text{ and } w_2>w_1;\\ ECI(G_1)\cup WCI(G_2)\\ \text{if } w_{t_1}+w_{t_2}<w_1=w_2.\end{cases}$$

(7) $CI(G[TS(G)])=CI(G_1[TS(G_1)])$.

(8) $CI(G) = WCI(G)$.

**Lemma 8.** Assume that $G$ is obtained from two disjoint distance hereditary graphs $G_1$ and $G_2$ by a true twin operation , and both $G_1$ and $G_2$ hold the strong duality.

$$(1)\ CT(G)=\begin{cases}\min\{CT(G_1[TS(G_1)]),CT(G_2[TS(G_2)])\}\\ \text{ if } w_{t_1}+w_{t_2}>\max\{w_1,w_2\};\\ \min\{SCT(G_1),WCT(G_1)\cup\\ CT(G_2[TS(G_2)])\}\\ \text{if } w_{t_1}+w_{t_2}=w_1>w_2;\\ \min\{SCT(G_2),WCT(G_2)\cup\\ CT(G_1[TS(G_1)])\}\\ \text{if } w_{t_1}+w_{t_2}=w_2>w_1;\\ \min\{SCT(G_1)\cup WCT(G_2),\\ WCT(G_1)\cup SCT(G_2)\}\\ \text{if } w_{t_1}+w_{t_2}=w_1=w_2;\\ CT(G_1)=WCT(G_1)\\ \text{if } w_{t_1}+w_{t_2}<w_1 \text{ and } w_1>w_2;\\ CT(G_2)=WCT(G_2)\\ \text{if } w_{t_1}+w_{t_2}<w_2 \text{ and } w_2>w_1;\\ WCT(G_1)\cup WCT(G_2)\\ \text{if } w_{t_1}+w_{t_2}<w_1=w_2.\end{cases}$$

(2) $CT(G[TS(G)])=\min\{CT(G_1[TS(G_1)]),$
$CT(G_2[TS(G_2)])$.

$$
(3)\; WCT(G) = \begin{cases} \phi & \text{if } w_{t_1} + w_{t_2} > \max\{w_1, w_2\}; \\ WCT(G_1) & \text{if } w_{t_1} + w_{t_2} = w_1 > w_2; \\ WCT(G_2) & \text{if } w_{t_1} + w_{t_2} = w_2 > w_1; \\ WCT(G_1) \cup WCT(G_2) & \text{if } w_{t_1} + w_{t_2} = w_1 = w_2; \\ WCT(G_1) & \text{if } w_{t_1} + w_{t_2} < w_1 \text{ and } w_1 > w_2; \\ WCT(G_2) & \text{if } w_{t_1} + w_{t_2} < w_2 \text{ and } w_2 > w_1; \\ WCT(G_1) \cup WCT(G_2) & \text{if } w_{t_1} + w_{t_2} < w_1 = w_2. \end{cases}
$$

$$
(4)\; SCT(G) = \begin{cases} \min\{CT(G_1[TS(G_1)]), CT(G_2[TS(G_2)])\} & \text{if } w_{t_1} + w_{t_2} > \max\{w_1, w_2\}; \\ \min\{SCT(G_1), WCT(G_1) \cup CT(G_2[TS(G_2)])\} & \text{if } w_{t_1} + w_{t_2} = w_1 > w_2; \\ \min\{SCT(G_2), WCT(G_2) \cup CT(G_1[TS(G_1)])\} & \text{if } w_{t_1} + w_{t_2} = w_2 > w_1; \\ \min\{SCT(G_1) \cup WCT(G_2), WCT(G_1) \cup SCT(G_2)\} & \text{if } w_{t_1} + w_{t_2} = w_1 = w_2; \\ \min\{SCT(G_1), WCT(G_1) \cup CT(G_2[TS(G_2)])\} & \text{if } w_{t_1} + w_{t_2} < w_1 \text{ and } w_1 > w_2; \\ \min\{SCT(G_2), WCT(G_2) \cup CT(G_1[TS(G_1)])\} & \text{if } w_{t_1} + w_{t_2} < w_2 \text{ and } w_2 > w_1; \\ \min\{SCT(G_1) \cup WCT(G_2), WCT(G_1) \cup SCT(G_2)\} & \text{if } w_{t_1} + w_{t_2} < w_1 = w_2. \end{cases}
$$

$$
(5)\; WCI(G) = \begin{cases} \phi & \text{if } w_{t_1} + w_{t_2} > \max\{w_1, w_2\}; \\ WCI(G_1) & \text{if } w_{t_1} + w_{t_2} = w_1 > w_2; \\ WCI(G_2) & \text{if } w_{t_1} + w_{t_2} = w_2 > w_1; \\ WCI(G_1) \cup WCI(G_2) & \text{if } w_{t_1} + w_{t_2} = w_1 = w_2; \\ WCI(G_1) & \text{if } w_{t_1} + w_{t_2} < w_1 \text{ and } w_1 > w_2; \\ WCI(G_2) & \text{if } w_{t_1} + w_{t_2} < w_2 \text{ and } w_2 > w_1; \\ WCI(G_1) \cup WCI(G_2) & \text{if } w_{t_1} + w_{t_2} < w_1 = w_2. \end{cases}
$$

$$
(6)\; ECI(G) = \begin{cases} TT(G) & \text{if } w_{t_1} + w_{t_2} > \max\{w_1, w_2\}; \\ WCI(G_1) \cup XT(G) & \text{if } w_{t_1} + w_{t_2} = w_1 > w_2; \\ WCI(G_2) \cup TX(G) & \text{if } w_{t_1} + w_{t_2} = w_2 > w_1; \\ WCI(G_1) \cup WCI(G_2) \cup XX(G) & \text{if } w_{t_1} + w_{t_2} = w_1 = w_2; \\ WCI(G_1) \cup XT(G) & \text{if } w_{t_1} + w_{t_2} < w_1 \text{ and } w_1 > w_2; \\ WCI(G_2) \cup TX(G) & \text{if } w_{t_1} + w_{t_2} < w_2 \text{ and } w_2 > w_1; \\ WCI(G_1) \cup WCI(G_2) \cup XX(G) & \text{if } w_{t_1} + w_{t_2} < w_1 = w_2. \end{cases}
$$

(7) $CI(G[TS(G)]) = TT(G)$.

$$
(8)\; CI(G) = \begin{cases} ECI(G) & \text{if } w_{t_1} + w_{t_2} \geq \max\{w_1, w_2\}; \\ WCI(G), & \text{otherwise.} \end{cases}
$$

**Theorem 2.** Distance hereditary graphs are maximum-clique perfect.

**Theorem 3.** For any distance hereditary graph $G$, $\tau_M(G)$ and $\alpha_M(G)$ can be computed in linear time.

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