The Local Distinguishing Number of Polygonal Prisms *

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Abstract

A *polygonal prism* with $2n$ vertices, denoted by PP_n, is a product graph of a C_n and a K_2. From the outward appearance, a polygonal prism PP_n consists of two base faces and n lateral faces. Each lateral face is an induced square, or a 4-cycle, of PP_n. To distinguish all of the lateral faces of PP_n, we color every vertex in the graph. Two lateral faces are distinguishable in colors if one face cannot be obtained from the other by rotating and flipping. As a result, the local distinguishing number problem with respect to a polygonal prism, or LDPP for short, is to determine the minimum number of colors required to distinguish each lateral face of the polygonal prism. The LDPP is a variation of graph coloring problem. In this paper, we propose a coloring algorithm and bound the local distinguishing number of a polygonal prism.

Keyword: graph coloring problem, distinguishing number problem, local distinguishing number problem, polygonal prism, product graph.

1 Introduction

The *distinguish number* of a graph G is the minimum number of colors for which there exists an assignment of colors to the vertices of G, so that the position of each vertex can be distinguished. In [1], a famous example about the distinguish number was introduced. The problem encountered when a key holder could not distinguish the keys on his/her key ring and wanted to mark the keys using the minimum number of colors. It has been shown that three colors are required when the key ring contain 3,4,5 keys. However, only two colors are needed when the number of keys is greater than 5.

In [3], the authors defined the i-th *local distinguishing number* of a cycle, or i-LDC for short, as a variation of the distinguish number problem. Suppose that every vertex in an n-cycle is colored, and each of the vertices stands for the center of an induced $(2i+1)$-path in the cycle. Then, the i-LDC problem is to determine the minimum number of colors required to distinguish all the different types of $(2i+1)$-paths in the cycle. An upper bound for the i-LDC problem is proposed in [3].

In this paper, we define a new type of local distinguishing number problem which is applied in a special class of graphs, called *polygonal prism*. A polygonal prism with $2n$ vertices, denoted by PP_n, is a product graph of C_n and K_2. For example, PP_6 is a product graph of C_6 and K_2, as shown in Figure 1.

A polygonal prism PP_n is also a planar graph. From the outward appearance, PP_n consists of two *base faces* and n *lateral faces*. Each lateral face is an induced square, or a 4-cycle, of PP_n, while each base face is an induced n-cycle of PP_n. To distinguish all of the lateral faces of PP_n, we have to color

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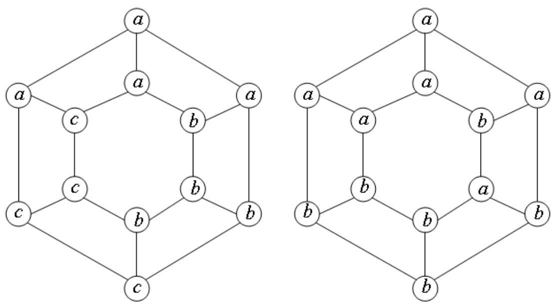

(a) (b)

Figure 1: An example of polygonal prism with 12 vertices, PP_6.

each vertex in the graph. Two lateral faces are distinguishable in colors if one face cannot be obtained from the other by rotating and flipping. As a result, the local distinguish number problem with respect to a polygonal prism, or LDPP for short, is to determine the minimum number of colors required to distinguish each lateral face of the polygonal prism. Using Figure 1 as an example, the color set used in Figure 1(a) and Figure 1(b) are a, b, c and a, b, respectively. The coloring of PP_6 is distinguishable in both Figure 1(a) and Figure 1(b). Since two colors are used in Figure 1(b), the local distinguishing number of PP_6 is two instead of three.

A polygonal prism PP_n contains n lateral faces. Each of the lateral faces is marked with four colored vertices and is denoted by a four-column tuple $[c_1, c_2, c_3, c_4]$, where c_1,c_2,c_3,c_4 are the color labels of the four vertices. Note that c_1 is chosen from any vertex of the lateral face and c_1,c_2,c_3,c_4 can appear in clockwise or anticlockwise order. For example, the six lateral faces of PP_6 in Figure 1(a) are $[a, a, b, a]$, $[a, b, b, b]$, $[b, b, c, b]$, $[b, c, c, c]$, $[c, c, a, c]$ and $[c, a, a, a]$. The representation of a lateral face in PP_n is not unique. For instance, the four lateral faces $[a, a, b, a]$, $[a, b, a, a]$, $[b, a, a, a]$, and $[a, a, a, b]$ are identical. By the definition of LDPP, we have the following property.

Property 1 *Let* $F_1 = [x_1, x_2, x_3, x_4]$ *and* $F_2 = [y_1, y_2, y_3, y_4]$ *be two lateral faces in* PP_n. F_1 *and* F_2 *are identical, or indistinguishable, if and only if (i) color sets* $\{x_1, x_3\} = \{y_1, y_3\}$ *and* $\{x_2, x_4\} = \{y_2, y_4\}$, *or (ii) color sets* $\{x_1, x_3\} = \{y_2, y_4\}$ *and* $\{x_2, x_4\} = \{y_1, y_3\}$.

In other words, lateral face F_1 can be obtained from lateral face F_2 by rotating and flipping if and only if condition (i) or (ii) holds. Consequently, color sets $\{x_1, x_3\}$ and $\{x_2, x_4\}$ can identify a lateral face $[x_1, x_2, x_3, x_4]$. Furthermore, if r colors are used, the maximum number of distinguishable lateral faces can be calculated.

Property 2 *Let* r *be the number of colors and* $r \geq 4$. *The maximum number of distinguishable lateral faces may be up to* $\frac{r^4+2r^3+3r^2+2r}{8}$.

If two colors are chosen from r colors and the two colors are not necessarily different each other, then there are $m = \binom{r+1}{2}$, or $\frac{r(r+1)}{2}$, different pairs of colors. Based on Property 1, the number of different ways to coloring a lateral face is $\binom{m+1}{2}$, or $\frac{r^4+2r^3+3r^2+2r}{8}$. For examples, if $r = 2$, then the maximum number of distinguishable lateral faces is 6. Figure 1(b) shows the case. If $r = 3$, then the maximum number of distinguishable lateral faces is 21. Using our coloring method, a PP_{21} with 21 distinguishable lateral faces can be constructed. However, in case of $r = 4$, although the maximum number of distinguishable lateral faces is 55, we can just construct a PP_{54} with 54 distinguishable lateral faces.

The LDPP is a variation of graph coloring problem. In this paper, we propose a coloring algorithm and bound the local distinguishing number of a polygonal prism. The difference between the maximum number and the number of lateral faces obtained from our coloring algorithm is $\binom{r}{4}$, or $\frac{r^4-6r^3+11r^2-6r}{12}$.

This paper is organized as follows. Section 2 proposes the main results. Section 3 proves the correctness of the results. Finally, section 4 gives the concluding remarks.

2 Main results

In this section, we shall show how to efficiently color PP_n and make its n lateral faces distinguishable.

Suppose r colors ($r \geq 4$) are used to color the four vertices of a lateral face in PP_n. There are four types of literal faces, i.e., 1-colored lateral faces, 2-colored lateral faces, 3-colored lateral faces, and 4-colored lateral faces. Clearly, we call a lateral face colored by s colors, $1 \leq s \leq 4$, an s-colored lateral face. The four types of literal faces are enumerated as follows:
(i) Choosing only one color x, then we can identify r distinguishable lateral faces, like $[x, x, x, x]$.
(ii) Choosing two colors x and y to color a face, there are four distinguishable lateral faces can be identified, i.e., $[x, x, y, x]$, $[x, x, y, y]$, $[x, y, x, y]$ and $[y, y, x, y]$. Thus, we can distinguish $\binom{r}{2}$ quartettes of lateral faces if two colors are chosen.
(iii) Choosing three colors x, y and z to color a face, there are six distinguishable lateral faces can be identified, i.e., $[x, y, z, x]$, $[x, y, x, z]$, $[y, x, z, y]$, $[y, x, y, z]$, $[z, x, y, z]$ and $[z, x, z, y]$. Thus, we can distinguish $\binom{r}{3}$ groups of six lateral faces if three colors are chosen.
(iv) Choosing four colors x, y, z and w to color a face, there are three distinguishable lateral faces can be identified, i.e., $[x, w, z, y]$, $[y, z, x, w]$ and $[x, y, w, z]$. Thus, we can distinguish $\binom{r}{4}$ groups of three lateral faces.

The total number of identified lateral faces is $r + 4\binom{r}{2}+6\binom{r}{3}+3\binom{r}{4}$, or $\frac{r^4+2r^3+3r^2+2r}{8}$, that is the same as what we have mentioned in Property 2. The purpose of the algorithm described in this section is to arrange these lateral faces to form a PP_n and make n as large as possible. Thus, we call the algorithm the **LFA** (Lateral Faces Arrangement) algorithm. The input of the LFA algorithm is the number of colors, while the output of the algorithm is a PP_n that satisfies the local distinguishing requirement.

The first step of the LFA algorithm is to arrange all the 1-colored and 2-colored lateral faces. We summarize the approach as follows:
(1) Construct a complete graph K_r.
(2) If r is even, then remove a maximum matching edge set from K_r to make it Eulerian, denoted by K_r^*.
(3) Find a Eulerian circuit in K_r or K_r^* such that the lateral faces can be arranged from $[x, x, x, x]$ to $[y, y, y, y]$, and so forth. For example, the Eulerian circuit in Figure 2 is (a, b, c, a). That is, we arrange the lateral faces from $[a, a, a, a]$ to $[b, b, b, b]$, to $[c, c, c, c]$, and then turn back to $[a, a, a, a]$. Similarly, the Eulerian circuit in K_4^* is (a, b, c, d, a), as shown in Figure 3.
(4) According to the Eulerian circuit, merge the 1-colored and 2-colored lateral faces using a template like these types: $[x, x, x, x]$, $[x, y, x, x]$, $[y, x, y, x]$, $[y, x, x, y]$, $[y, y, x, y]$ and $[y, y, y, y]$. See Figure 2 for an example of $r = 3$.
(5) In case that r is even, the 2-colored lateral faces related to the removed edge set in K_r are not arranged yet. If an edge (x, z) is removed in step (2), we insert all lateral faces related to (x, z) into the position between $[x, x, x, x]$ and $[x, y, x, x]$ using a template like these types: $[x, z, x, x]$, $[z, x, z, x]$, $[z, z, x, z]$ and $[x, x, z, z]$. For example, all the 2-colored lateral faces related to (a, c) and (b, d) are inserted into a feasible positions as shown in Figure 3.

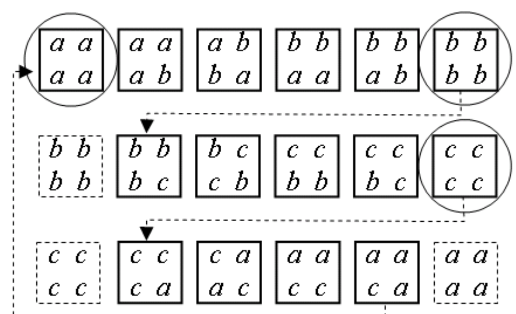

Figure 2: Arrange lateral faces identified in conditions(i) and (ii) for $r = 3$.

Next, the LFA algorithm deals with the 3-colored lateral faces if the number of colors is greater than two. Every group of six lateral faces colored by x, y and z can be properly inserted into a feasible posi-

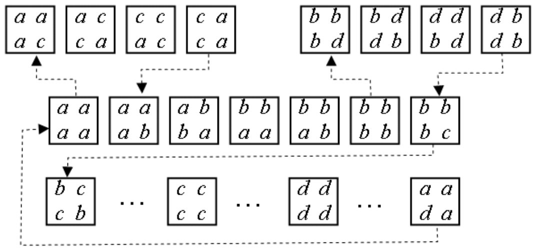

Figure 3: Arrange lateral faces identified in conditions(i) and (ii) for $r = 4$.

tion between $[x, y, x, x]$ and $[y, x, y, x]$. This insertion is done by using a template like these types: $[z, x, y, x]$, $[y, x, x, z]$, $[z, y, x, y]$, $[z, x, y, z]$, $[y, z, x, z]$ and $[y, x, z, y]$. Besides, the lateral faces between $[y, x, z, y]$ and $[y, y, y, y]$ should be flipped in order to meet the coloring requirement. See Figure 4 for an explanation. Notice that, after the insertion, another group of six lateral faces colored by x, y and w can be inserted into a position between $[x, y, x, x]$ and $[z, x, y, x]$.

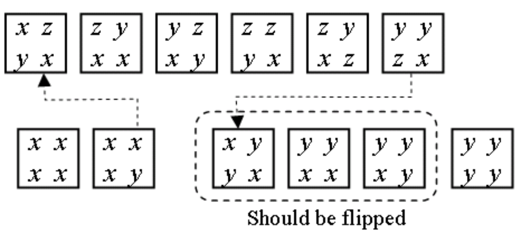

Figure 4: Insert the lateral faces identified in condition (iii).

Finally, the LFA algorithm takes the 4-colored lateral faces into considerations. Although there are three lateral faces are identified, only two of them, i.e., $[y, z, x, y]$ and $[x, w, z, y]$, are suitable for inserting into a position between $[w, x, *, *]$ and $[*, *, x, w]$ (where * stands for any color of x, y, z and w). The lateral faces between $[x, w, z, y]$ and $[*, *, t, t]$ should be flipped before the insertion. See Figure 5 for an explanation.

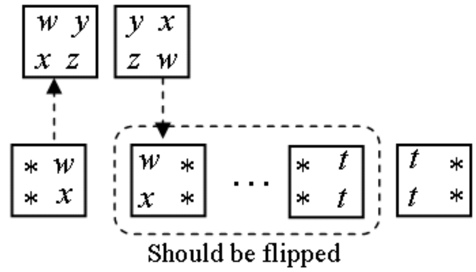

Figure 5: Insert the lateral faces identified in condition (iv).

3 Correctness proofs

In this section, we show that the LFA algorithm can be applied to color all PP_n for $n \geq 21$, and that if r colors are used in the LFA algorithm, then r is an upper bound of the LDPP.

Let $f(r)$ be the number of literal faces obtained from the LFA algorithm using r colors ($r \geq 4$). From the explanation of the above section, we have the following lemma.

Lemma 3 *For all $r \geq 4$, $f(r) = \frac{r^4+6r^3-r^2+6r}{12}$, and r is an upper bound of the local distinguishing number of $PP_{f(r)}$.*

Proof. The value of $f(r)$ is the summation of the number of lateral faces which could be arranged in conditions (i), (ii), (iii) and (iv). That is, $f(r) = r + 4\binom{r}{2} + 6\binom{r}{3} + 2\binom{r}{4}$. This formula can be reduced to $\frac{r^4+6r^3-r^2+6r}{12}$.

Clearly, $PP_{f(r)}$ is the largest possible polygonal prism that the LFA algorithm can obtain using r colors. In property 2, we have learned that the maximum number of lateral faces distinguished by r colors ($r \geq 4$) is $\frac{r^4+2r^3+3r^2+2r}{8}$, which is greater than $f(r)$. Thus, the local distinguishing number of $PP_{f(r)}$ may be less then or equal to r. $\square$

The LFA algorithm can be applied to color all PP_n

for $n \geq 21$, and make all of the n lateral faces distinguishable. This work is done by removing some lateral faces from $PP_{f(r)}$ and keeping the remains distinguishable.

Lemma 4 *If $f(r) - 2\binom{r}{4} \leq n < f(r)$ and $r \geq 4$, then r bounds the local distinguishing number of PP_n.*

Proof. The if condition can be expressed with another form, i.e., either $n = f(r) - 2i$ or $n = f(r) - 2i + 1$ with $1 \leq i \leq \binom{r}{4}$.

In case of $n = f(r) - 2i$, PP_n can be obtained by removing i pairs of the 4-colored lateral faces from $PP_{f(r)}$. Each pair of the 4-colored lateral faces looks like $[y, z, x, w]$ and $[x, w, z, y]$, which have been inserted into the $PP_{f(r)}$ in the LFA algorithm. We just reverse the procedure and thus keep n lateral faces distinguishable.

In case of $n = f(r) - 2i + 1$, PP_n can be obtained by removing $i - 1$ pairs of the 4-colored lateral faces and one 1-colored lateral face from $PP_{f(r)}$. A 1-colored lateral face looks like $[x, x, x, x]$.

Since r is an upper bound of $PP_{f(r)}$, if $f(r-1) \leq n$, r is an upper bound of local distinguishing number of PP_n. $\square$

Lemma 5 *If $f(r) - 2\binom{r}{4} - 6\binom{r}{3} \leq n < f(r) - 2\binom{r}{4}$ and $r \geq 4$, then r bounds the local distinguishing number of PP_n.*

Proof. The if condition can be rewritten as follows: $n = f(r) - 2\binom{r}{4} - 3j$ or $n = f(r) - 2\binom{r}{4} - 3j + 1$ or $n = f(r) - 2\binom{r}{4} - 3j + 2$, where $1 \leq j \leq 2\binom{r}{3}$.

In case of $n = f(r) - 2\binom{r}{4} - 3j$, PP_n can be obtained by removing $\binom{r}{4}$ pairs of the 4-colored lateral faces and j groups of three 3-colored lateral faces from $PP_{f(r)}$. Let x, y, z be the chosen colors. Each group of the 3-colored lateral faces looks like $[y, x, x, z]$, $[z, x, y, z]$, $[y, x, z, y]$, or $[z, x, y, x]$, $[y, z, x, z]$, $[x, y, z, y]$. Note that the former three lateral faces are removed in advance, as shown in Figure 6.

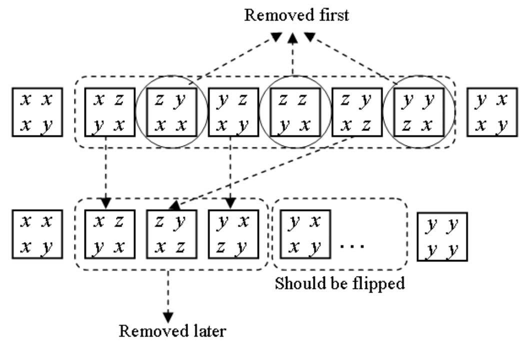

Figure 6: The procedure of removing 3-colored lateral faces.

In case of $n = f(r) - 2\binom{r}{4} - 3j + 1$, PP_n is obtained by removing $\binom{r}{4}$ pairs of the 4-colored lateral faces, $j - 1$ groups of three 3-colored lateral faces and two 1-colored lateral faces from $PP_{f(r)}$.

In case of $n = f(r) - 2\binom{r}{4} - 3j + 2$, PP_n is be obtained by removing $\binom{r}{4}$ pairs of the 4-colored lateral faces, $j - 1$ groups of three 3-colored lateral faces and one 1-colored lateral faces from $PP_{f(r)}$.

Since r is an upper bound of $PP_{f(r)}$, if $f(r-1) \leq n$, r is an upper bound of local distinguishing number of PP_n. $\square$

Lemma 6 *If $f(r) - 2\binom{r}{4} - 6\binom{r}{3} - r \leq n < f(r) - 2\binom{r}{4} - 6\binom{r}{3}$ and $r \geq 4$, then r bounds the local distinguishing number of PP_n.*

Proof. The if condition can be rewritten as $n = f(r) - 2\binom{r}{4} - 6\binom{r}{3} - k$ with $1 \leq k \leq r$. In this case, PP_n can be obtained by removing all of the 4-colored and 3-colored lateral faces, and k 1-colored lateral faces from $PP_{f(r)}$.

Since r is an upper bound of $PP_{f(r)}$, if $f(r-1) \leq n$, r is an upper bound of local distinguishing number of PP_n. $\square$

We summarize Lemmas 4, 5, 6 and 7 as the following theorem.

Theorem 7 *For all n, $f(r-1) < n \leq f(r)$ and $r \geq 4$, r bounds the local distinguishing number of PP_n.*

Proof. The only condition that we do not take into consideration is $f(r-1) \leq n < f(r) - 2\binom{r}{4} - 6\binom{r}{3} - r$, or $f(r-1) < n < 4\binom{r}{2}$. Since $f(r-1)$ is greater than $4\binom{r}{2}$ for $r \geq 5$, we only have to consider the case of $r = 4$.

When all 4-colored literal faces absent in the LFA algorithm, we can make 21 literal faces distinguishable. That is, $f(3) = 21$. Based on the LFA algorithm and Lemma 6, we can make 24 2-colored literal faces distinguishable when 4 colors are used. Thus, in case of $r = 4$, we only have to consider the cases of $n = 22$ and 23.

A pair of 2-colored lateral faces like $[x, y, x, x]$ and $[y, x, y, x]$ can be merged into a lateral face $[y, x, x, x]$ and keep it distinguishable. Consequently, PP_{22} and PP_{23} are obtained from PP_{24} by merging lateral faces.

Since r is an upper bound of $PP_{f(r)}$ and, for all $f(r-1) < n \leq f(r)$ and $r \geq 4$, we can make the n lateral faces of PP_n distinguishable using r colors, r is an upper bound of the LDPP. $\square$

4 Concluding remarks

We have proved that the LFA algorithm can bound the local distinguishing number of a PP_n for n ≥ 21. As a matter of fact, for $3 \leq n < 21$, the LDPP has been solved by an enumerating method that is similar to the LFA algorithm. The local distinguishing number of PP_n is 2 for $3 \leq n \leq 6$, while the local distinguishing number of PP_n is 3 for $7 \leq n \leq 21$.

The local distinguishing number problems are interesting and could be applied to other graph classes with different styles. They could have applications on electronic keys, object identification, and so on.

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