

Hamiltonian laceability on the edge fault star graph ^{*}

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Abstract

The star graph is an attractive alternative to the hypercube graph. It possess many nice topological properties. Edge fault tolerance is an important issue for a network since the edges in the network may fail sometimes. In this paper, we show that the n -dimensional star graph is $(n - 3)$ -edge fault tolerant hamiltonian laceable, $(n - 3)$ -edge fault tolerant strongly hamiltonian laceable, and $(n - 4)$ -edge fault tolerant hyper hamiltonian laceable. All these results are optimal in a sense described in this paper.

Keywords: star graph, hamiltonian laceable, strongly hamiltonian laceable, hyper hamiltonian laceable, fault tolerant.

1 Introduction

Network topology is a crucial factor for a network since it determines the performance of the network. For convenience of discussing their properties, networks are usually represented by graphs. In this paper, a network topology is represented

by a simple undirected graph, which is loopless and without multiple edges. For the graph definition and notation we follow [5]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a, b) \mid a, b \in V \text{ and } a \neq b\}$, where (a, b) denotes an unordered pair. We call V the *vertex set* and E the *edge set*. Vertices a and b are adjacent if and only if $(a, b) \in E$. A path is a sequence of adjacent vertices, denoted by $\langle v_0, v_1, \dots, v_k \rangle$, in which v_0, v_1, \dots, v_k are distinct except that possibly $v_0 = v_k$. The length of the path is k . For ease of description, we may use P or $\langle v_0, P, v_k \rangle$ to denote the path. A hamiltonian path of G is a path which crosses all vertices of G . A graph G is hamiltonian connected if there exists a hamiltonian path joining any two vertices of G .

Hypercubes [14] and stars [1] are bipartite graphs. A graph $G = (V_0 \cup V_1, E)$ is *bipartite* if $V_0 \cap V_1 = \emptyset$ and $E \subseteq \{(a, b) \mid a \in V_0 \text{ and } b \in V_1\}$. Given vertices x and y , we say that x and y are in the same partite set if $x, y \in V_i$ or in different partite sets if $x \in V_i$ and $y \in V_{1-i}$ for $i \in \{0, 1\}$. However, the concept of hamiltonian connectivity does not apply to bipartite graphs because bipartite graphs are definitely not hamil-

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tonian connected except for a few exceptions such as K_2 or K_1 . As such a property is important, Wong [22] introduced the concept of *hamiltonian laceability* on bipartite graphs. A bipartite graph $G = (V_0 \cup V_1, E)$ is hamiltonian laceable if there is a hamiltonian path between any two vertices x and y which are in different partite sets. It is trivial that $|V_0|$ must be equal to $|V_1|$. On the condition of $|V_0| = |V_1|$, Hsieh et al. [11] extended this concept and proposed the concept of *strongly hamiltonian laceability*. G is strongly hamiltonian laceable if it is hamiltonian laceable and there is a path of length $|V_0| + |V_1| - 2$ between any two vertices in the same partite set. Lewinter et al. [15] introduced another concept, *hyper hamiltonian laceability*. G is hyper hamiltonian laceable if it is hamiltonian laceable and for any vertex $v \in V_i$, there is a hamiltonian path of $G - v$ between any two vertices in V_{1-i} . So hyper hamiltonian laceability is definitely also strongly hamiltonian laceability.

Fault tolerance is an important property of network performance. Hsieh, Chen, and Ho [10] proposed *the edge fault-tolerant hamiltonicity* to measure the performance of the hamiltonian property in the faulty networks. A graph G is *k-edge-fault tolerant hamiltonian* if $G - F$ remains hamiltonian for every $F \subseteq E(G)$ with $|F| \leq k$. Extending this concept, we introduce the following indicators. *The edge fault tolerant hamiltonian laceability* of the graph G is the integer value f such that for any $F \subseteq E(G)$ with $|F| \leq f$, $G - F$ is still hamiltonian laceable and there exists $F' \subseteq E(G)$ with $|F'| = f + 1$ such that $G - F'$ is not hamiltonian laceable. We use $eftHHL(G)$ to denote this capacity. Similarly, we can define *the edge fault tolerant strongly hamiltonian laceability* of G , denoted by $eftSHL(G)$, and *the edge*

fault tolerant hyper hamiltonian laceability of G , denoted by $eftHHL(G)$. $eftSHL(G)$ is the integer f such that for any $F \subseteq E(G)$ with $|F| \leq f$, $G - F$ is still strongly hamiltonian laceable and there exists $F' \subseteq E(G)$ with $|F'| = f + 1$ such that $G - F'$ is not strongly hamiltonian laceable. $eftHHL(G)$ is the integer f such that for any $F \subseteq E(G)$ with $|F| \leq f$, $G - F$ is still hyper hamiltonian laceable and there exists $F' \subseteq E(G)$ with $|F'| = f + 1$ such that $G - F'$ is not hyper hamiltonian laceable. We say a graph G is optimal with respect to $eftHHL$ ($eftSHL$, $eftHHL$, respectively) if for a fixed number of vertices, G contains the least number of edges among all graphs G' with $eftHHL(G') = eftHHL(G)$ ($eftSHL(G') = eftSHL(G)$, $eftHHL(G') = eftHHL(G)$).

This paper is to study these three indicators of the star graphs. The star graphs [2] are Cayley graphs. They have many nice properties such as recursiveness, vertex and edge symmetry, maximal fault tolerance, sublogarithmic degree and diameter [2]. These properties are important for designing interconnection topologies for parallel and distributed systems. Star graphs are able to embed cycles, grids [12], trees [3], and hypercubes [18]. Many efficient communication algorithms for shortest-path routing [19], multiple-path routing [6], broadcasting [17], gossiping [4], and scattering [8] were proposed. And many efficient algorithms designed for sorting and merging [16], selection [19], Fourier transform [7], and computational geometry [20] have been proposed. As a result, star graphs are recognized as an attractive alternative to the hypercubes.

In this paper, we show that the n -dimensional star graphs are optimal with respect to the edge fault tolerant hamiltonian laceability, the edge

fault tolerant strongly hamiltonian laceability, and the edge fault tolerant hyper hamiltonian laceability. In the next section, we introduce the definition of star graphs. And then in Section 3, we show our main result. Finally, we make our conclusion in Section 4.

2 Definition and Basic Properties

In this section, we introduce the definition and some properties of the star graph.

Definition 1 *The n -dimensional star graph is denoted by S_n . The vertex set V of S_n is $\{a_1 \cdots a_n \mid a_1 \cdots a_n \text{ is a permutation of } 1, 2, \dots, n\}$ and the edge set E is $\{(a_1 a_2 \cdots a_{i-1} a_i a_{i+1} \cdots a_n, a_i a_2 \cdots a_{i-1} a_1 a_{i+1} \cdots a_n) \mid a_1 \cdots a_n \in V \text{ and } 2 \leq i \leq n\}$.*

By definition, S_n contains $n!$ vertices and each vertex is of degree $(n - 1)$. For example, vertex 1234 in S_4 connects to 2134, 3214, and 4231. S_1 , S_2 , and S_3 are a vertex, an edge, and a cycle of length 6, respectively. We show S_4 in Fig. 1. It is easy to observe that there are four vertex-disjoint S_3 's embedded in S_4 . The following proposition states this property.

Proposition 1 *There are $\frac{n!}{k!}$ vertex-disjoint S_k 's embedded in S_n for $k \geq 1$.*

Proof. Let $B = \{b_{k+1} \cdots b_n \mid b_{k+1} \cdots b_n \text{ is a permutation of any } (n - k) \text{ elements of } 1, 2, \dots, n\}$. So $|B| = \frac{n!}{k!}$. For any $b_{k+1} \cdots b_n \in B$, let $S_k^{b_{k+1} \cdots b_n}$ denote the induced subgraph of S_n with vertex set $\{a_1 \cdots a_n \mid a_{k+1} \cdots a_n = b_{k+1} \cdots b_n\}$. Obviously, $S_k^{b_{k+1} \cdots b_n}$ and $S_k^{b'_{k+1} \cdots b'_n}$ are vertex-disjoint for $b_{k+1} \cdots b_n \neq b'_{k+1} \cdots b'_n$ and $V(S_n) = \bigcup_{b_{k+1} \cdots b_n \in B} V(S_k^{b_{k+1} \cdots b_n})$.

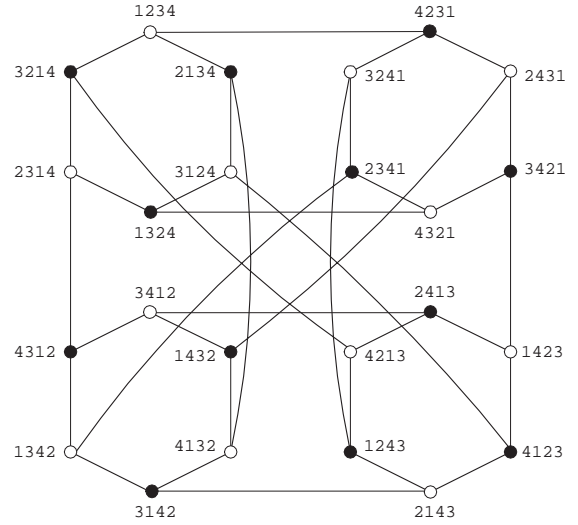


Figure 1: 4-dimensional star graph.

Next, we construct an isomorphism mapping vertices in each $S_k^{b_{k+1} \cdots b_n}$ to vertices in S_k . Let $u = u_1 \cdots u_k b_{k+1} \cdots b_n$ be some vertex in $S_k^{b_{k+1} \cdots b_n}$. For each vertex $a = a_1 \cdots a_k b_{k+1} \cdots b_n$, define $f_u^k(a_1 \cdots a_k b_{k+1} \cdots b_n) = i_1 \cdots i_k$, where $a_j = u_{i_j}$ and $1 \leq j \leq k$, e.g., if the first digit of a is equal to the l th digit of u , then the first digit of $f_u^k(a)$ is l . For example, let $u = 54123$ be a vertex in S_5^{123} . Then $f_u^2(54123) = 12$ and $f_u^2(45123) = 21$. We can easily check that $\{f_u^k(v) \mid v \in V(S_k^{b_{k+1} \cdots b_n})\} = V(S_k)$ and $(f_u^k(v_1), f_u^k(v_2))$ is an edge if and only if (v_1, v_2) is an edge. So $S_k^{b_{k+1} \cdots b_n} \cong S_k$ and the proposition follows. \square

In the following discussion, we will frequently use the notation $S_k^{b_{k+1} \cdots b_n}$ defined in the proof above. We call $S_k^{b_{k+1} \cdots b_n}$ a *substar* of S_n or specifically, a *k -dimensional substar* of S_n . Let u be a vertex not in $S_k^{b_{k+1} \cdots b_n}$. We say that u is adjacent to $S_k^{b_{k+1} \cdots b_n}$ if u is adjacent to a vertex in $S_k^{b_{k+1} \cdots b_n}$. And we call $S_k^{b_{k+1} \cdots b_n}$ an *adjacent substar* of u . The following proposition and corollary are concerning adjacent substars:

Proposition 2 Given k with $1 \leq k \leq n-1$ and $b_{k+1} \cdots b_n$, a vertex $u = u_1 \cdots u_n$ is adjacent to $S_k^{b_{k+1} \cdots b_n}$ if and only if $u_{k+1} \cdots u_{i-1} u_{i+1} \cdots u_n = b_{k+1} b_{k+2} \cdots b_n$ for some i with $k+1 \leq i \leq n$.

Proof. As u is not in $S_k^{b_{k+1} \cdots b_n}$, $u_{k+1} \cdots u_n \neq b_{k+1} \cdots b_n$. However, there is a neighbor $u_i u_2 \cdots u_{i-1} u_{i+1} \cdots u_n$ of u in $S_k^{b_{k+1} \cdots b_n}$ for $2 \leq i \leq n$. Thus, $k+1 \leq i \leq n$ and $u_{k+1} \cdots u_{i-1} u_{i+1} \cdots u_n = b_{k+1} b_{k+2} \cdots b_n$. \square

Corollary 1 There are $(k-1)!$ edges between $S_k^{b_{k+1} \cdots b_n}$ and $S_k^{b'_{k+1} \cdots b'_n}$ if there is exactly one different bit between $b_{k+1} \cdots b_n$ and $b'_{k+1} \cdots b'_n$.

Proof. Without loss of generality, assume that $b_{k+1} \neq b'_{k+1}$ and $b_{k+2} \cdots b_n = b'_{k+2} \cdots b'_n$. The first bit of all vertices in $S_k^{b_{k+1} \cdots b_n}$ being adjacent to $S_k^{b'_{k+1} \cdots b'_n}$ must be b'_{k+1} . So the number of these vertices is $(k-1)!$. And the corollary follows. \square

For example, there are $(n-2)!$ edges between S_{n-1}^i and S_{n-1}^j for $1 \leq i \neq j \leq n$. We use $E^{i,j}(S_n)$ to denote the set of these edges. And we call these edges outgoing edges of S_{n-1}^i (or S_{n-1}^j). Particularly, we say (u, v) an outgoing edge of u if $(u, v) \in E^{i,j}(S_n)$ for some $1 \leq i \neq j \leq n$.

It has been shown that the star graphs are edge symmetric [2], i.e., for any two edges $(x, y), (u, v) \in E(S_n)$, there is an automorphism of S_n mapping x, y into u, v , respectively. For ease of description, we use $\pi(F)$ to denote the edge set $\{(\pi(u), \pi(v)) \mid (u, v) \in F\}$ if π is an automorphism of S_n and $F \subseteq S_n$. Thus, we have following proposition.

Proposition 3 Let $F \subseteq E(S_n)$. Then there is an edge set $F' \subseteq E(S_n)$ and an automorphism π

of S_n such that $\pi(F) = F'$ and $|F' \cap E(S_{n-1}^i)| \leq |F| - 1$ for each $1 \leq i \leq n$.

Proof. If $|F \cap E(S_{n-1}^i)| \leq |F| - 1$ for each $1 \leq i \leq n$, let F' be F and π be the identity mapping. Then the statement follows. Otherwise, choose an arbitrary edge $(x, y) \in F$. With the edge symmetric property, there is an automorphism π of S_n such that $\pi(x) = 123 \cdots (n-1)n$ and $\pi(y) = n23 \cdots (n-1)1$. Let $F' = \pi(F)$. So $(123 \cdots (n-1)n, n23 \cdots (n-1)1) \in F'$. But $(123 \cdots (n-1)n, n23 \cdots (n-1)1) \notin E(S_{n-1}^i)$ for all $1 \leq i \leq n$. Thus, $|F' \cap E(S_{n-1}^i)| \leq |F| - 1$ for each $1 \leq i \leq n$. \square

By this proposition, given any edge set $F \subseteq E(S_n)$, we may assume that $|F \cap E(S_{n-1}^i)| \leq |F| - 1$ for each $1 \leq i \leq n$. This property will help us simplify the proof a lot.

3 Main result

In this section, we present our main result on the three indicators, which are the edge fault tolerant hamiltonian laceability (*eftHLL*), edge fault tolerant strongly hamiltonian laceability (*eftSHL*), and edge fault tolerant hyper hamiltonian laceability (*eftHHL*) of the star graphs. We provide a lemma to give three upper bounds for the general graphs and then three theorems to give the exact values for the three indicators on the star graphs. We will see that all the values match the upper bounds. So the star graphs are optimal with respect to these properties. Now we show the upper bounds.

Lemma 1 Let $G = (V_0 \cup V_1, E)$ be a bipartite graph with $|V_0| = |V_1|$ and let δ be the minimum degree of G among all vertices. We have $eftHLL(G) \leq \delta - 2$, $eftSHL(G) \leq \delta - 2$ for $\delta \geq 2$, and $eftHHL(G) \leq \delta - 3$ for $\delta \geq 3$.

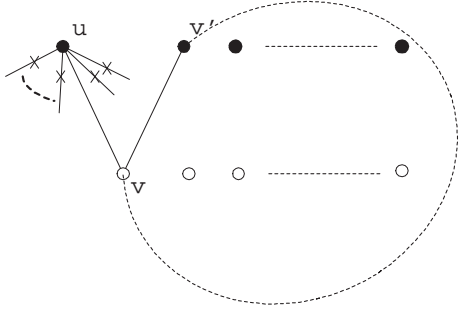


Figure 2: Upper bound for $eftHL(G)$.

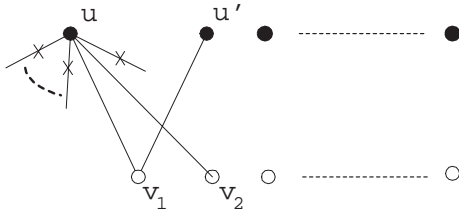


Figure 3: Upper bound for $eftHHL(G)$.

Proof. Assume that the degree of vertex u is δ . Removing $(\delta - 1)$ edges connecting to u results in the isolation of u . Suppose that v is the remainder vertex connecting to u and v' is a neighbor of v which is not u (see Fig. 2). Then it is easy to check that there is no hamiltonian path from v to v' . So G is at most $(\delta - 2)$ -edge fault tolerant hamiltonian laceable and obviously, at most $(\delta - 2)$ -edge fault tolerant strongly hamiltonian laceable.

Then consider removing $(\delta - 2)$ edges which connect to u . Suppose that v_1 and v_2 are the remainder vertices connecting to u and let u' be a vertex connecting to v_1 which is not u (see Fig. 3). Then it is easy to check that there is no hamiltonian path of $G - u'$ from v_1 to v_2 . So G is at most $(\delta - 3)$ -edge fault tolerant hyper hamiltonian laceable.

Hence, the lemma follows. \square

Next, we show the capacity of star graphs on these three indicators. First, we use a computer program to check the base case S_4 (see Fig. 1) and the case indeed holds for S_4 . So we state the results in the following lemma. Then we prove our results by induction.

Lemma 2 S_4 is 1-edge fault tolerant hamiltonian laceable, 1-edge fault tolerant strongly hamiltonian laceable, and hyper hamiltonian laceable.

To make the proofs clear, we introduce the following transform:

Definition 2 Given a fixed n , let $V \subseteq \{1, 2, \dots, n\}$ and $F \subseteq E(S_n)$. Then $STG_n(V, F)$ is the graph $G(V, E)$ such that $E = \{(i, j) \mid i, j \in V \text{ and } E^{i,j}(S_n) \cap F < \frac{(n-2)!}{2}\}$. (STG means to transmit a star graph to another graph.)

In fact, STG_n maps the substar S_{n-1}^i in $(S_n - F)$ into the vertex i in G for all $i \in V$. And for $i \neq j \in V$, if i and j are adjacent in G , there is a vertex in each partite set of S_{n-1}^i adjacent to S_{n-1}^j in $(S_n - F)$. So we have following lemma:

Lemma 3 Let $G = STG_n(V, F)$ for $V \subseteq \{1, 2, \dots, n\}$ with $|V| \geq 2$ and $F \subseteq E(S_n)$. And let $x \in S_{n-1}^{j_1}$ and $y \in S_{n-1}^{j_2}$ with $j_1 \neq j_2 \in V$ such that x, y are in different partite sets. Assume that $S_{n-1}^i - F$ is hamiltonian laceable for each $i \in V$. Then there is a path from x to y crossing all vertices in all S_{n-1}^i for $i \in V$ without crossing edges in F if there is a hamiltonian path from j_1 to j_2 in G .

Proof. Let $|V| = h$. And let $\langle j_1, j_3, j_4, \dots, j_h, j_2 \rangle$ be a hamiltonian path from j_1 to j_2 in G . Since j_1 and j_3 are adjacent in G , we can find a vertex $v^1 \in V(S_{n-1}^{j_1})$ adjacent to $S_{n-1}^{j_3}$ such that v^1, x are in different partite

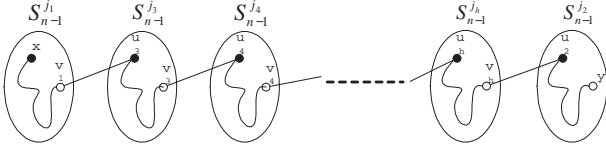


Figure 4: Remaining path.

sets and the outgoing edge, say (v^1, u^3) , of v^1 is not in F (see Fig. 4). Similarly, we can find vertices $v^3 \in S_{n-1}^{j_3}, v^4 \in S_{n-1}^{j_4}, \dots, v^h \in S_{n-1}^{j_h}$ adjacent to $S_{n-1}^{j_4}, S_{n-1}^{j_5}, \dots, S_{n-1}^{j_2}$, respectively, such that $v^1, v^3, v^4, \dots, v^h$ are in the same partite set and the outgoing edges of these vertices are not in F . Assume that the outgoing edges of these vertices are $(v^3, u^4), (v^4, u^5), \dots, (v^h, u^2)$. Then by assumption that each $S_{n-1}^i - F$ is hamiltonian laceable, we can construct a path from x to y crossing all vertices in all S_{n-1}^i for $i \in V$ as follows:

$$\langle x, \dots, v^1, u^3, \dots, v^3, u^4, \dots, v^h, u^2, \dots, y \rangle.$$

Hence, the lemma follows. \square

Now we can show our first result:

Theorem 1 S_n is $(n - 3)$ -edge fault tolerant hamiltonian laceable for $n \geq 4$.

Proof. We prove it by induction. By Lemma 2, we know that S_4 is 1-edge fault tolerant hamiltonian laceable. In the induction step, we assume that S_{n-1} is $(n - 4)$ -edge fault tolerant hamiltonian laceable for $n \geq 5$. Then consider S_n .

Let $F \subseteq E(S_n)$ be arbitrary faulty edge set such that $|F| \leq n - 3$. By Proposition 3, we may assume that $|F \cap E(S_{n-1}^i)| \leq n - 4$ for each $1 \leq i \leq n$. So $S_{n-1}^i - F$ is still hamiltonian laceable for each $1 \leq i \leq n$. Let $x \in V(S_{n-1}^{j_1})$ and $y \in V(S_{n-1}^{j_2})$ such that x, y are in different partite sets. We

shall construct a fault-free hamiltonian path from x to y . Consider the following two cases:

Case 1. $j_1 \neq j_2$. Let $V = \{1, 2, \dots, n\}$. Since $|F| \leq n - 3 < \frac{(n-2)!}{2}$ for $n \geq 5$, $E^{i,j}(S_n) \cap F < \frac{(n-2)!}{2}$ for any $i \neq j \in V$. So $G = STG_n(V, F)$ is a complete graph. It is easy to find a hamiltonian path of G from j_1 to j_2 . By Lemma 3, there is a hamiltonian path of S_n from x to y .

Case 2. $j_1 = j_2 = j$. There is a hamiltonian path P of S_{n-1}^j from x to y . The length of P is $(n - 1)! - 1$. So we can find an edge, say (u, v) , on path P such that the outgoing edges of u and v are fault-free. (If such (u, v) does not exist, $|F| \geq \frac{(n-1)!-1}{2} > n - 3$ for $n \geq 5$.) Let $P = \langle x, P_1, u, v, P_2, y \rangle$ and $(u, v'), (v, u')$ are the outgoing edges of u and v , where $v' \in S_{n-1}^{j_3}$ and $u' \in S_{n-1}^{j_4}$ (see Fig. 5). v' and u' are in different partite sets of S_n and $j_3 \neq j_4$. Let $V = \{1, 2, \dots, n\} - j$. Then $G = STG_n(V, F)$ is a complete graph with $(n - 1)$ vertices since $\frac{(n-2)!}{2} > |F|$ for $n \geq 5$. Thus, there is a hamiltonian path of G from j_3 to j_4 . By Lemma 3, there is a path P_3 crossing all vertices of all S_{n-1}^i for $i \in V$ from v' to u' . So a hamiltonian path of S_n from x to y can be constructed as follows:

$$\langle x, P_1, u, v', P_3, u', v, P_2, y \rangle$$

Hence, the theorem follows. \square

Since S_n is $(n - 1)$ regular, by Lemma 1, S_n is optimal with respect to edge fault tolerant hamiltonian laceability and $eftHLL(S_n) = n - 3$.

Theorem 2 S_n is $(n - 3)$ -edge fault tolerant strongly hamiltonian laceable for $n \geq 4$.

Proof. We also prove it by induction. S_4 is shown to be 1-edge fault tolerant strongly hamiltonian laceable in Lemma 2. So we need only to

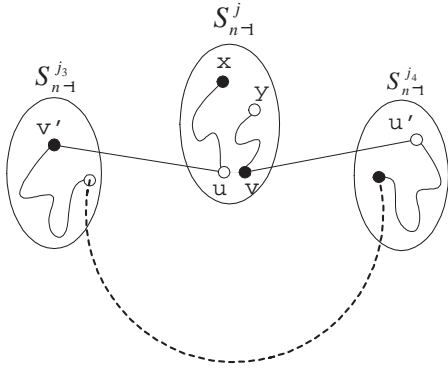


Figure 5: x and y are in the same substar.

consider the induction step. Assume that S_{n-1} is $(n-4)$ -edge fault tolerant strongly hamiltonian laceable for $n \geq 5$ and consider S_n .

Given any fault edge set F in S_n with $|F| \leq n-3$, by Proposition 3, we can assume that $|F \cap E(S_{n-1}^i)| \leq n-4$ for each $1 \leq i \leq n$. So $S_{n-1}^i - F$ is still strongly hamiltonian laceable for each $1 \leq i \leq n$. Let $x \in V(S_{n-1}^{j_1})$ and $y \in V(S_{n-1}^{j_2})$ such that x and y are in the same partite set. Consider the following two cases:

Case 1. $j_1 \neq j_2$. Let V_e be the number of vertices which are in the different partite set from x and which are not adjacent to $S_{n-1}^{j_2}$. Then V_e is equal to $\frac{(n-1)!}{2} - \frac{(n-2)!}{2}$ which is strictly greater than $|F|$. So there is a fault-free edge (u^1, v^3) , i.e., not in F , such that $u^1 \in V(S_{n-1}^{j_1})$, $v^3 \in V(S_{n-1}^{j_3})$ for $j_3 \notin \{j_1, j_2\}$, and x, y, u^1 are in the same partite set of S_n (see Fig. 6). By the induction hypothesis, there is a path P_1 of length $(n-1)!-2$ in $S_{n-1}^{j_1}$ from x to u^1 . Then consider the remainder subgraphs. Let $V = \{1, 2, \dots, n\} - \{j_1\}$. Thus, $|V| \geq 2$ and $G = STG_n(V, F)$ is a complete graph. There is a hamiltonian path of G from j_3 to j_2 . Since u^1 and y are in the same partite set, y and v^3 are in different partite sets. So there is a path P_2

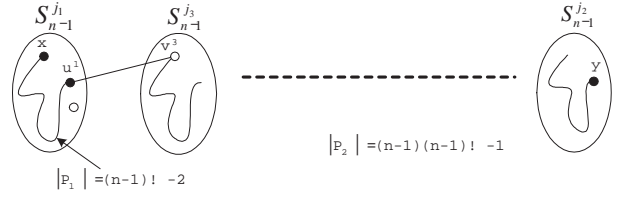


Figure 6: x and y are in different substars.

crossing all vertices of all S_{n-1}^i for $i \in V$ from v^3 to y . The length of this path is $(n-1)(n-1)!-1$. We can construct a path from x to y as follows:

$$\langle x, P_1, u^1, v^3, P_2, y \rangle$$

The length of this path is

$$[(n-1)!-2] + 1 + [(n-1)(n-1)!-1] = n!-2$$

So the theorem follows in this case.

Case 2. $j_1 = j_2 = j$. The proof of this case is similar to that of case 2 in Theorem 1 except that the path in S_{n-1}^j from x to y is of length $(n-1)!-2$.

Hence, the theorem follows. \square

Since S_n is $(n-1)$ regular, by Lemma 1, S_n is also optimal with respect to the edge fault tolerant strongly hamiltonian laceability.

Theorem 3 S_n is $(n-4)$ -edge fault tolerant hyper hamiltonian laceable for $n \geq 4$.

Proof. The proof is a little more complex than the previous two theorems. Again, S_4 is hyper hamiltonian laceable by Lemma 2. So we show that the statement is true for $n \geq 5$. Assume that S_{n-1} is $(n-5)$ -edge fault tolerant hyper hamiltonian laceable for $n \geq 5$.

Let F be a faulty edge set in S_n with $|F| \leq n-4$. By Proposition 3, we may assume that

$|F \cap E(S_{n-1}^i)| \leq n - 5$ for each $1 \leq i \leq n$. So $S_{n-1}^i - F$ is still hyper hamiltonian laceable and obviously, strongly hamiltonian laceable for each $1 \leq i \leq n$. Given a vertex v , in the following we will construct a hamiltonian path of $(S_n - F) - v$ between any two vertices in the partite set which v is not in. Let x and y be two such vertices. Consider the following four cases:

Case 1. v, x, y are in the same substar, say $S_{n-1}^{j_1}$ (see Fig. 7(a)). By the induction hypothesis, there is a hamiltonian path P of $(S_{n-1}^{j_1} - F) - v$ from x to y . The length of P is $(n-1)! - 2 > 2|F|$ for $n \geq 5$. So there is an edge (u^1, v^1) on P such that the outgoing edges of u^1 and v^1 , say (u^1, v^2) and (v^1, u^3) , are fault-free. (x, u^1 are not necessary in the same partite set.) Let $P = \langle x, P_1, u^1, v^2, P_2, y \rangle$. Clearly, v^2 and u^3 are in different partite sets of S_n . Assume that $v^2 \in S_{n-1}^{j_2}$ and $u^3 \in S_{n-1}^{j_3}$. So $j_2 \neq j_3$. Let $V = \{1, 2, \dots, n\} - \{j_1\}$. Then $STG_n(V, F)$ is a complete graph. There is a hamiltonian path from j_2 to j_3 and so a path P_3 from v^2 to u^3 crossing all vertices of S_{n-1}^i for all $i \in V$. Therefore, we can construct a hamiltonian path of $(S_n - F) - v$ as: $\langle x, P_1, u^1, v^2, P_3, u^3, v^1, P_2, y \rangle$.

Case 2. $v, x \in S_{n-1}^{j_1}$ and $y \in S_{n-1}^{j_2}$ with $j_1 \neq j_2$ (see Fig. 7(b)). Let $j_3 \neq j_2$. Since $\frac{(n-2)!}{2} - 1 > |F|$ for $n \geq 5$, we can easily find a vertex $u^1 \neq x \in S_{n-1}^{j_1}$ such that u^1 and x are in the same partite set and the outgoing edge of u^1 , say (u^1, v^3) , is fault-free. (Note that since $u^1 \neq x$, there are $\frac{(n-2)!}{2} - 1$ choices for u^1 in $S_{n-1}^{j_1}$.) By the induction hypothesis, there is a hamiltonian path P_1 of $(S_{n-1}^{j_1} - F) - v$ from x to u^1 . Let $V = \{1, 2, \dots, n\} - \{j_1\}$. Then $STG_n(V, F)$ is a complete graph. Note that v^3 and y are in different partite sets. So there is a hamiltonian path

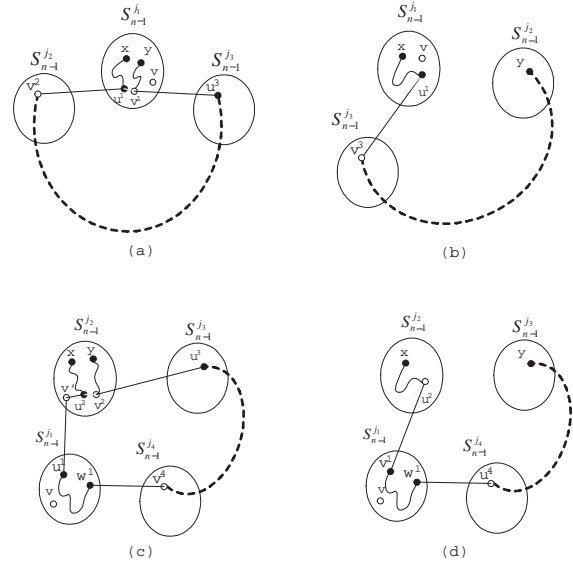


Figure 7: Edge fault tolerant hyper hamiltonian laceability of the star.

from j_3 to j_2 and a path P_2 from v^3 to y crossing all vertices of all S_{n-1}^i for $i \in V$. Hence, we have a hamiltonian path $\langle x, P_1, u^1, v^3, P_2, y \rangle$ of $(S_n - F) - v$.

Case 3. $v \in S_{n-1}^{j_1}$ and $x, y \in S_{n-1}^{j_2}$ with $j_1 \neq j_2$ (see Fig. 7(c)). Since $\frac{(n-2)!}{2} > |F|$, there is a vertex $v' \in V(S_{n-1}^{j_2})$ adjacent to $S_{n-1}^{j_1}$ such that the outgoing edge of v' , say (v', u^1) , is fault-free and v', v are in the same partite set. By the induction hypothesis, there is a hamiltonian path P of $(S_{n-1}^{j_2} - F) - v'$ from x to y . Since there are $(n-2)$ neighbors of v' in $S_{n-1}^{j_2}$ and $|F| < (n-2)$, there exists an edge (u^2, v^2) on P such that u^2 is adjacent to v' and the outgoing edge of v^2 , say (v^2, u^3) , is fault-free. Clearly, $j_3 \notin \{j_1, j_2\}$ since v^2, v' are neighbors of u^2 but $v^2 \neq v'$. Let $P = \langle x, P_1, u^2, v^2, P_2, y \rangle$. (Note that P may be $\langle x, P_1, v^2, u^2, P_2, y \rangle$ and the argument of this case is similar to the following discussion.) Let $j_4 \notin \{j_1, j_2, j_3\}$. Since $\frac{(n-2)!}{2} - 1 > |F|$ for $n \geq 5$, there is a vertex $w^1 \in S_{n-1}^{j_1}$ adjacent

to $S_{n-1}^{j_4}$ such that w^1, u^1 are in the same partite set and the outgoing edge of w^1 , say (w^1, v^4) , is fault-free. So $v^4 \in V(S_{n-1}^{j_4})$ and v^4, u^3 are in different partite sets. By the induction hypothesis, there is a hamiltonian path P_3 of $(S_{n-1}^{j_1} - F) - v$ from u^1 to w^1 . Let $V = \{1, 2, \dots, n\} - \{j_1, j_2\}$. Then $STG_n(V, F)$ is a complete graph. There is a hamiltonian path from j_4 to j_3 and so a path P_4 crossing all vertices of S_{n-1}^i for all $i \in V$ from v^4 to u^3 . Thus, we have a hamiltonian path of $(S_n - F) - v$ as follows:

$$\langle x, P_1, u^2, v^1, u^1, P_3, w^1, v^4, P_4, u^3, v^2, P_2, y \rangle.$$

Case 4. $v \in S_{n-1}^{j_1}$, $x \in S_{n-1}^{j_2}$, and $y \in S_{n-1}^{j_3}$ for distinct j_1, j_2 , and j_3 (see Fig. 7(d)). Since $\frac{(n-2)!}{2} > |F|$, there is a vertex $u^2 \in V(S_{n-1}^{j_2})$ adjacent to $S_{n-1}^{j_1}$ such that u^2, x are in different partite sets and the outgoing edge of u^2 , say (u^2, v^1) , is fault-free. By the induction hypothesis, there is a hamiltonian path P_1 of $(S_{n-1}^{j_2} - F)$ from x to u^2 . Let $j_4 \notin \{j_1, j_2, j_3\}$. In $S_{n-1}^{j_1}$, since $\frac{(n-2)!}{2} - 1 > |F|$, there is a vertex $w^1 \neq v^1$ adjacent to $S_{n-1}^{j_4}$ such that w^1, v^1 are in the same partite set and the outgoing edge of w^1 , say (w^1, u^4) , is fault-free. By the induction hypothesis, there is also a hamiltonian path P_2 of $(S_{n-1}^{j_1} - F) - v$ from v^1 to w^1 . For the remaining substars, let $V = \{1, 2, \dots, n\} - \{j_1, j_2\}$. Then $G = STG_n(V, F)$ is a complete graph. So there is a hamiltonian path of G from j_4 to j_3 and thus, a path P_3 from u^4 to y crossing all vertices of S_{n-1}^i for all $i \in V$. Finally, we have a hamiltonian path $\langle x, P_1, u^2, v^1, P_2, w^1, u^4, P_3, y \rangle$ of $(S_n - F) - v$.

Hence, the theorem follows. \square

Since S_n is $(n-1)$ regular, by Lemma 1, S_n is optimal with respect to the edge fault tolerant hyper hamiltonian laceability.

4 Conclusion

Fault tolerance is an important research subject of the multi-process computer systems. Graphs are usually used to represent the interconnection architecture of these systems, where vertices represent processors and edges represent links between processors. Many researches concerned the vertex-fault tolerant or edge-fault tolerant properties of some specific graphs. In this paper, we study some fault tolerant results of the star graphs. We show that the n -dimensional star graph is $(n-3)$ -edge fault tolerant hamiltonian laceable, $(n-3)$ -edge fault tolerant strongly hamiltonian laceable, and $(n-4)$ -edge fault tolerant hyper hamiltonian laceable

In particular, we use computer programs to check the base cases. It not only gives us some preliminary intuition but also simplifies our proof. If we did such check by theoretical proof, we would have spent too much effort since there would have been too many subcases to deal with. Apparently, such a method may be applied in other cases nowadays, especially, for those facts which can be proved by induction.

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