Construction for fault-tolerant Hamiltonian and Hamiltonian-connected graphs

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Abstract

The Hamiltonian property is one of major requirements in designing the topology of networks. Fault tolerance is also required for distributed systems. In this paper, we introduce the concepts of (node, edge) Hamiltonian-connectivity. Furthermore, we present construction schemes for fault-tolerant Hamiltonian and Hamiltonian-connected networks.

Keywords: fault tolerance, Hamiltonian cycle, Hamiltonian-connectivity, edge-Hamiltonian-connectivity, node-Hamiltonian-connectivity.

1、Introduction

The architecture of an interconnection network is always represented by a graph. We use graphs and networks interchangeably. The ring and linear array topologies have been used frequently due to their good properties such as simplicity, expandability and easiness of implementation. Fault tolerance is also important in parallel systems that have a relatively high probability of failure. The embedding of rings and linear arrays into known interconnection networks such as pancake graphs, faulty hypercubes, stars, arrangement graphs, and meshes and tori has been addressed in the literature[1-3, 6-7, 9, 11]. However, the longest rings or linear arrays embedding in hypercube or star networks with faulty nodes may not contain all fault-free nodes. Specially, the hypercube or star networks are not Hamiltonian networks with a faulty node.

Let G=(V, E) be a graph where V denotes the node set and E denotes the edge set of G. A *path* is a sequence of nodes such that two consecutive nodes are adjacent. A *Hamiltonian path* of G is a path whose nodes are distinct and span V. A graph is *Hamiltonian-connected* if there exist Hamiltonian paths between every two distinct nodes of this graph. A Hamiltonian-connected graph is equivalent a network which contains linear arrays for every distinct nodes. A *Hamiltonian cycle* is a cycle who traverses every node of V exactly once. A *Hamiltonian graph* is a graph that contains a Hamiltonian cycle. A graph G is *k-Hamiltonian* if G-F is Hamiltonian $\forall\ F \subset (V \cup E)$, $|F| \le k$. A graph G is *k-node-Hamiltonian* (*k-edge-Hamiltonian*) if G-F_n(G-F_e) is Hamiltonian $\forall\, F_n \subset V$($\forall\, F_e \subset E$) and $|F_n|$($|F_e|$) $\le$ k. A *k-Hamiltonian* graph is equivalent to a network that contains a ring with k faults.

In [12], Wang et al constructed a 3-regular, 1-Hamiltonian graph. Hung et al presented another 3-regular, 1-Hamiltonian graph in [4]. These graphs are both Hamiltonian-connected. A construction scheme for 1-Hamiltonian graph is introduced in [13]. In [5], the authors introduced a construction scheme for ($k+2$)-regular and k-Hamiltonian graphs. In this paper, we introduce Hamiltonian-connectivity for the fault tolerance of Hamiltonian-connected graph and present the construction scheme for graphs with fault tolerance for Hamiltonian-connected and Hamiltonian graph.

The rest of this paper is organized as follows. In Section 2, we introduce concept and examples of Hamiltonian-connectivity, node-Hamiltonian-connectivity, and edge-Hamiltonian-connectivity. Section 3 is devoted to the proofs of the fault tolerance Hamiltonian properties. Section 4 concludes this paper.

2、Hamiltonian-connectivity, node-Hamiltonian-connectivity and edge-Hamiltonian-connectivity

In this section, we will define Hamiltonian-connectivity, node-Hamiltonian-connectivity, and edge-Hamiltonian-connectivity and illustrate these concepts with some examples.

The *node-Hamiltonian-connectivity* of a graph G, denoted as $\kappa_h(G)$, is the minimum number of cardinality of $F_n \subset V$ such that G-F_n is not Hamiltonian-connected. The *edge-Hamiltonian-connectivity* of G, denoted as $\lambda_h(G)$, is min $\{|F_e| \mid G - F_e$ is not Hamiltonian-connected $\forall\ F_e \subset E\}$. The *Hamiltonian-connectivity* of G, denoted by $\sigma_h(G)$, is the minimum cardinality of $F \subset (V \cup E)$ such that G-F is not Hamiltonian-connected. Obviously, $\sigma_h(G) \le \kappa_h(G)$ and $\sigma_h(G) \le \lambda_h(G)$ for every graph G.

Let $\kappa(G)$ and $\lambda(G)$ be the node-connectivity and edge-connectivity of the graph G, respectively. Applying these definitions, we can obtain that $\kappa_h(G) \le \kappa(G)$ and $\lambda_h(G) \le \lambda(G)$ for any graph G. In [18], Whitney presents $\kappa(G) \le \lambda(G)$ for any graph G. But the relation $\kappa_h(G) \le \lambda_h(G)$ does not hold for every graph G. The complete graph K_n is a counter example. $\kappa_h(K_n) = \kappa(K_n) = n - 1 > n - 3 = \lambda_h(K_n)$. On the other hand, $\kappa_h(H_1) = 1 \le 3 = \lambda_h(H_1)$ since $H_2 = H_1 - \{c\}$ has no Hamiltonian path between nodes a and b as illustrated in Figure 1.

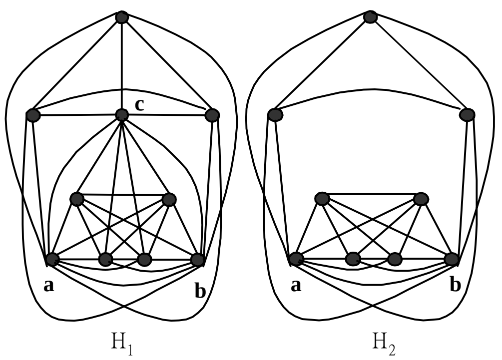


Figure 1 $\kappa_h(H_1) = 1 \quad , \lambda_h(H_1) = 3$.

3. Construction schemes for fault tolerance of Hamiltonian and Hamiltonian-connected graphs

In this section, we will prove some theorems about Cartesian product and Hamiltonian properties. Applying these theorems, we can obtain the construction schemes for the k-Hamiltonian (or k-node-Hamiltonian, k-edge-Hamiltonian) graph G that $\sigma_h(G)$ (or $\kappa_h(G)$, $\lambda_h(G)$) = k.

Let $G=(V_1,E_1)$ and $H=(V_2,E_2)$ be a graph. The graph $G \times H$ is the Cartesian product of the graphs G and H. The node set $V(G \times H)$ is $\{(u,i) \mid \forall\ u \in V_1$ and $i \in V_2\}$. The edge set $E(G \times H)$ is $\{((u,i),(v,j)) \mid$ if $(u,v) \in E_1$, $i=j$ or $u=v$, $(i,j) \in E_2\}$. Thus $V(G \times K_2)=\{(v,i) \mid \forall\ v \in V$ and $i=1, 2\}$ and $E(G \times K_2)=\{((u,i),(v,j)) \mid$ if $(u,v) \in E$, $i=j$ or $u=v$, $i \ne j\}$. Let the degree of v in G, denoted by $deg_G(v)$, be the number of edges in G incident to v. Let G_i be a sub-graph of $G \times K_2$ and G_i isomorphic to G. To be specific, the node set of G_i is $\{(v,i) \mid \forall\ v \in V\}$ and the edge set of G_i is $\{((u,i),(v,i)) \mid \forall\ (u,v) \in E\}$.

Let F, F_1 and F_2 denote sets of faulty components including faulty nodes and edges in $G \times K_2$, G_1 and G_2, respectively. F_3 is a set of faulty components including faulty edges of two corresponding nodes. That is, $F_3 = F \cap \{((u,1),(u,2)) \mid \forall u \in V\}$. And $F = F_1 \cup F_2 \cup F_3$.

Let HC and HP be a Hamiltonian cycle and Hamiltonian path in $G \times K_2$, respectively. Let HC_i be a Hamiltonian cycle and $HP_i((u,i),(v,i))$ be a Hamiltonian path between (u,i) and (v,i) in G_i for $i=1, 2$. The distance between nodes u and v, denoted by $d(u\ v)$, is the length of a shortest path from u to v. Let $(u,1)$ and $(u,2)$ be two corresponding nodes. Let (f,i) be a fault of F_i. Let $<(v,i) \xrightarrow{f} (u,j)>$ denote the path $<(v,i) \rightarrow f \rightarrow (u, j)>$ if f is a faulty node, or denote the path $<(v, i) \rightarrow (u, j)>$ if f is a faulty edge.

Theorem 1 *Let $G=(V, E)$ be a $(k+2)$-regular, k-Hamiltonian graph and $\sigma_h(G) = k$. The graph $G \times K_2$ is a $(k+3)$-regular, $(k+1)$-Hamiltonian graph and $\sigma_h(G \times K_2) = k+1$.*

Proof: Since $\deg_{G \times K_2}(v,i) = \deg_G(v)+1$, $G \times K_2$ is $(k+3)$-regular. We first prove that $(G \times K_2)$-F is a Hamiltonian graph in which $F \subset G \times K_2$ and

$|F|=k+1$. We prove this theorem by the following cases.

Case 1: $|F_1|=k+1$ or $|F_2|=k+1$, without loss of generality, we assume that $|F_1|=k+1$.

Since G_1 is a k-Hamiltonian graph, $G_1 - (F - \{f\})$ contains a Hamiltonian cycle HC_1 for some $f\in F$. Thus G_1-F contains a Hamiltonian path $HP_1((a,1),(b,1)) = HC_1 - \{f\}$, for some $(a,1)$, $(b,1)\in V(G_1)$. Since the $\sigma_h(G_2)=k$, there exists a Hamiltonian path $HP_2((a,2),(b,2))$ in G_2. Hence, we can construct a Hamiltonian cycle $\langle (a,1) \xrightarrow{HP_1((a,1),(b,1))} (b,1) \rightarrow (b,2) \xrightarrow{HP_2((b,2),(a,2))} (a,2) \rightarrow (a,1) \rangle$ of $(G\times K_2)$-F, as illustrated in Figure 2.

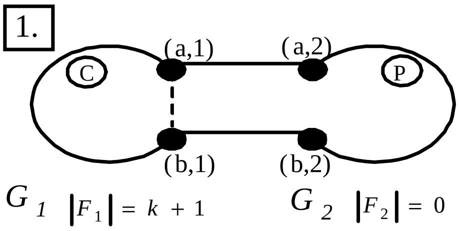


Figure 2 The Hamiltonian cycle of $(G\times K_2)$-F.

Case 2: $|F_1|\leq k$ and $|F_2|\leq k$

Since $|V(G)|\geq k+3$, there exist $(a,1)\in V(G_1)$ and $(a,2)\in V(G_2)$ where $(a,1)$,$(a,2)$, and $((a,1),(a,2))\notin F$. Since G_i is k-Hamiltonian, there exist $k+2-|F_i|$ edges incident to (a,i) on some Hamiltonian cycles of G_i for $i=1, 2$. Furthermore, $(k+2-|F_1|)+(k+2-|F_2|)=2k+4-|F_1|-|F_2|=k+3+|F_3|$. Thus there exist $1+|F_3|$ neighbors (v_r,i) for $1\leq r\leq 1+|F_3|$, of (a,i) such that $((a,i),(v_r,i))$ is on some Hamiltonian cycles of G_i, for $i=1, 2$. Therefore, there exists a node $(v_r,2)$ for $((v_r,1),(v_r,2))\notin F$ where $((a,1),(v_r,1))$ and $((a,2),(v_r,2))$ are on some Hamiltonian cycles HC_1 and HC_2 of G_1 and G_2, respectively. Hence, a Hamiltonian cycle can be constructed as follows: $HC_1 \cup HC_2 \cup \{((a,1),(a,2)), ((v_r,1),(v_r,2))\}-\{((a,1),(v_r,1)),((a,2),(v_r,2))\}$ of $(G\times K_2)$-F, as illustrated in Figure 3.

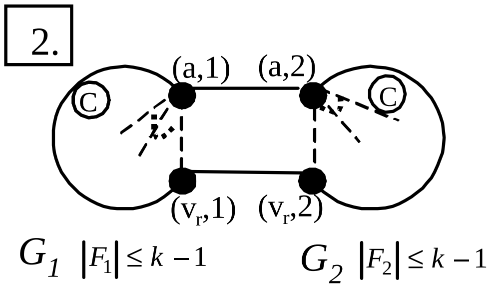


Figure 3 The Hamiltonian cycle of $(G\times K_2)$-F.

According to the above proof, we can obtain that $G\times K_2$ is $(k+1)$-Hamiltonian.

In the following, we will show that for all $((s,i),(d,j))\in V(G\times K_2)$, $1\leq i, j\leq 2$, there exists a Hamiltonian path HP$((s,i),(d,j))$ from (s,i) to (d,j) in $(G\times K_2)$-F for $F\subset G\times K_2$ and $|F|=k$.

Case 3: $|F_1|=k$ or $|F_2|=k$, without loss of generality, we can assume $|F_1|=k$.

Case 3.1: $i=1, j=1$

Since the $\sigma_h(G_1)=k$, G_1-$(F$-$\{(f)\})$ contains a Hamiltonian path $\langle (s,1) \rightarrow \cdots \rightarrow (a,1) \xrightarrow{f} (b,1) \rightarrow \cdots \rightarrow (d,1) \rangle$. Thus G_1-F forms two paths $P((s,1),(a,1))$ and $Q((b,1),(d,1))$. Since G_2 is Hamiltonian-connected, there exists a Hamiltonian path $HP_2((a,2),(b,2))$ in G_2. Hence, we can construct a Hamiltonian path $\langle (s,1) \xrightarrow{P((s,1),(a,1))} (a,1) \rightarrow (a,2) \xrightarrow{HP_2((a,2),(b,2))} (b,2) \rightarrow (b,1) \xrightarrow{Q((b,1),(d,1))} (d,1) \rangle$ of $(G\times K_2)$-F, as illustrated in Figure 4.

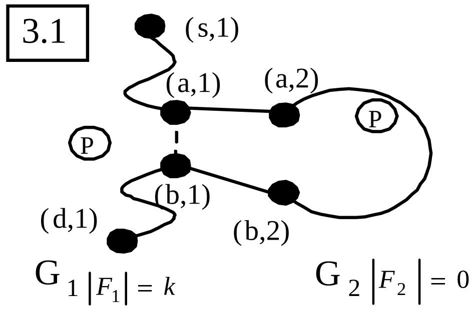


Figure 4 The Hamiltonian path from $(s,1)$ to $(d,1)$ of $(G\times K_2)$-F.

Case 3.2: $i=1, j=2$

Since G_1 is k-Hamiltonian, there exists a Hamiltonian cycle HC_1 in G_1-F_1. Let $((a,1),(s,1))$ and $((b,1),(s,1))$ be an edge of HC_1. One of $(a,1)$,$(b,1)$ must not be identical to $(d,1)$.Without loss of generality, we may assume $(a,1) \neq (d,1)$. Since G_2 is Hamiltonian-connected, there exists a Hamiltonian path $HP_2((a,2),(d,2))$ in G_2. Hence, a Hamiltonian path from $(s,1)$ to $(d,2)$ can be constructed as follows: HC_1-$\{(a,1),(s,1)\} \cup \{((a,1),(a,2))\} \cup HP_2((a,2),(d,2))$, as illustrated in Figure 5.

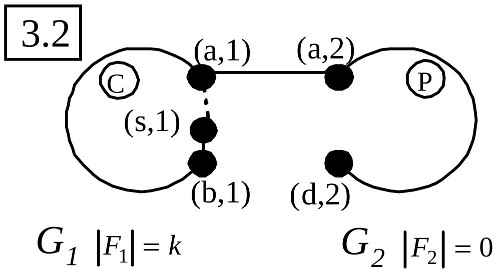


Figure 5 The Hamiltonian path from $(s,1)$ to $(d,2)$ of $(G\times K_2)$-F.

Case 3.3: $i=2, j=2$

Since G is simple graph, $|V(G)| \geq k+3$. Thus, there exist $(a,1) \in V(G_1)$ and $(a,2) \in V(G_2)$ where $(a,1),(a,2)$, $((a,1),(a,2)) \notin F$ and $(a,2) \neq (s,2) \neq (d,2)$. Since G_1 is k-Hamiltonian, there exist $k+2$-$|F_1|$ edges incident to $(a,1)$ on some Hamiltonian cycles. Since $\sigma_h(G_2)=k$, there exist $k+1$ edges incident to $(a,2)$ on some Hamiltonian paths. Furthermore, $(k+2-|F_1|)+(k+1)=2k+3-|F_1|=k+3$. Therefore, there exists a fault-free neighbor (v_r,i) of $((v_r,1),(v_r,i)) \notin F$ where $((a,1),(v_r,1))$ and $((a,2),(v_r,i))$ are on some Hamiltonian cycle HC_1 of G_1 and some Hamiltonian path $HP_2((s,2),(d,2))$ of G_2, respectively. Hence we can construct a Hamiltonian path from $(s,2)$ to $(d,2)$ $HC_1 \cup HP_2((s,2),(d,2)) \cup \{((a,1),(a,2)), ((v_r,1),(v_r,2))\}$-$\{((a,1),(v_r,1)),((a,2),(v_r,2))\}$ of $(G\times K_2)$-F, as illustrated in Figure 6.

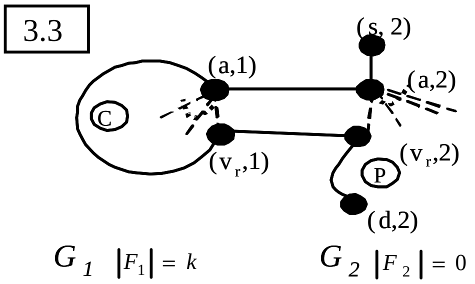


Figure 6 The Hamiltonian path from $(s,2)$ to $(d,2)$ of $(G\times K_2)$-F.

Case 4: $|F_1| \leq k-1$ and $|F_2| \leq k-1$

Case 4.1 $i=j$, without loss of generality, we can assume $i=1, j=1$.

Let $(a,1)$ be the vertex of $V(G_1)$ satisfying that $(a,1),(a,2)$ and $((a,1),(a,2)) \notin F$ and $(a,1) \neq (s,1), (a,1) \neq (d,1)$. Since the $\sigma_h(G_1)=k$, there exist $k+1$-$|F_1|$ edges incident to $(a,1)$ on some Hamiltonian path in G_1. Since $|F_1|+|F_2| \leq k$, there exist $k+2$-$|F_2| \geq 2+|F_1|$ neighbors of $(a,2)$ are fault-free. Since $k+1$-$|F_1|+k+2$-$|F_2| \geq k+3$, there exists a fault-free neighbor (v_s,i) of (a,i) such that $((a,i),(v_s,i))$ is on some Hamiltonian path $HP_1((s,1),(d,1))$ of G_1-F_1. Furthermore, G_2-F_2 is Hamiltonian-connected, there exists a Hamiltonian path $HP_2((a,2),(v_s,2))$. Thus, we can construct a Hamiltonian path from (s,1) to (d,1) $HP_1((s,1),(d,1))$-$\{((a,1),(v_s,1))\} \cup \{((a,1),(a,2))\} \cup \{((v_s,1),(v_s,2))\} \cup HP_2((a,2),(v_s,2))$, as illustrated in Figure 7.

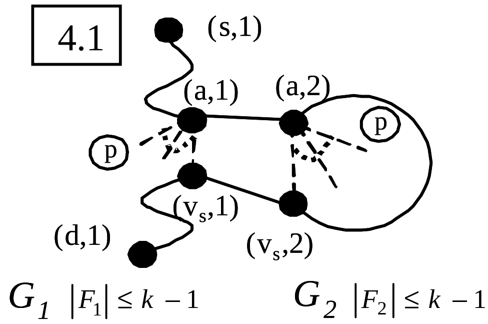


Figure 7 The Hamiltonian path from $(s,1)$ to $(d,1)$ of $(G\times K_2)$-F.

Case 4.2 $i \neq j$, without loss of generality, we can assume $i=1, j=2$.

There exists a node $(a,1) \in V(G_1)$ in which $(a,1) \neq (s,1), (a,1) \neq (d,1)$ and $(a,1)$, $(a,2)$ and $((a,1),(a,2))$ are fault-free. Since $|F_1|<k$ and $|F_2|<k$, G_1-F_1 and G_2-F_2 are Hamiltonian-connected. Let $HP_1((s,1),(a,1))$ and $HP_2((a,2),(d,2))$ denote the Hamiltonian paths of G_1 and G_2, respectively. Therefore, $G\times K_2$-F has a Hamiltonian path from $(s,1)$ to $(d,2)$. $\langle (s,1) \xrightarrow{HP_1((s,1),(a,1))} (a,1) \rightarrow (a,2) \xrightarrow{HP_2((a,2),(d,2))} (d,2) \rangle$ of $(G\times K_2)$-F, as illustrated in Figure 8.
This theorem is proved. □

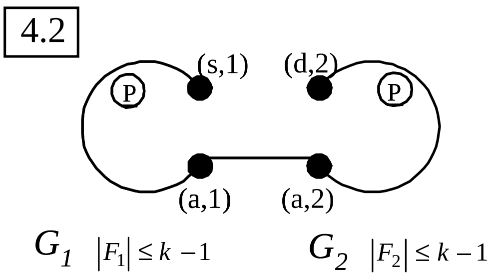


Figure 8 The Hamiltonian path from $(s,1)$ to $(d,2)$ of $(G\times K_2)$-F.

Theorem 2 *Let G=(V, E) be a (k+2)-regular, k-edge-Hamiltonian graph and* $\lambda_h(G)=k$. *The graph* $G\times K_2$ *is a (k+3)-regular, (k+1)-node-Hamiltonian graph and* $\lambda_h(G\times K_2)=k+1$.

The proof of Theorem 2 is very similar to Theorem 1. We skip the proof of this theorem.

Let F_n, F_{n_1} and F_{n_2} denote sets of faulty nodes in $G\times K_2$, G_1 and G_2, respectively. Thus $F_n = F_{n_1} \cup F_{n_2}$.

Theorem 3 *Let G=(V, E) be a (k+2)-regular, k-node-Hamiltonian graph and* $\kappa_h(G)=k$. *The graph* $G\times K_2$ *is a (k+3)-regular, (k+1)-node-Hamiltonian graph and* $\kappa_h(G\times K_2)=k+1$.

Proof: Since $\deg_{G\times K_2}(v,i)=\deg_G(v)+1$, $G\times K_2$ is a $(k+3)$-regular. We first show that $(G\times K_2)-F_n$ is a Hamiltonian graph in which $F_n\subset V(G\times K_2)$ and $|F_n|=k+1$. We prove this by the following cases.

Case 1: $|F_{n_1}|=k+1$ or $|F_{n_2}|=k+1$. Without loss of generality, we assume that $|F_{n_1}|=k+1$.

Since G_1 is a k-node-Hamiltonian graph, $G_1-(F_n-\{f\})$ contains a Hamiltonian cycle HC_1 for $f\in F_n$. Thus G_1-F_n contains a Hamiltonian path $HP_1((a,1),(b,1))=HC_1-\{f\}$, for $(a,1),(b,1)\in V(G_1)$. There exists a Hamiltonian path $HP_2((b,2),(a,2))$ in G_2 since G_2 is Hamiltonian-connected. Hence, we can construct a Hamiltonian cycle $\langle (a,1) \xrightarrow{HP_1((a,1),(b,1))} (b,1) \rightarrow (b,2) \xrightarrow{HP_2((b,2),(a,2))} (a,2) \rightarrow (a,1) \rangle$ of $(G\times K_2)-F_n$, as illustrated in Figure 9.

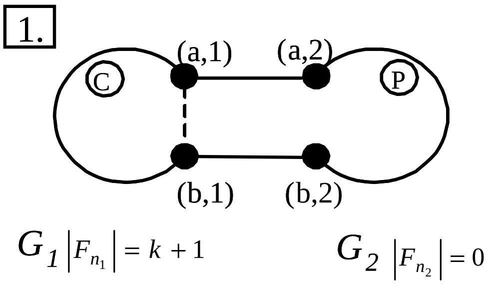


Figure 9 The Hamiltonian cycle of $(G\times K_2)-F_n$.

Case 2: $1\le|F_{n_1}|\le k$ and $1\le|F_{n_2}|\le k$

Case 2.1: $k=1$, that is $|F_{n_1}|=1$ and $|F_{n_2}|=1$.

Since both G_1 and G_2 are 1-node- Hamiltonian, there exist Hamiltonian cycle HC_1 and HC_2 of G_1 and G_2, respectively. Let $(a,1)\in V(G_1)$ and $(a,2)\in V(G_2)$ be two fault-free nodes. There exist edges $((a,i),(b,i))\in E(G_i)$ on the Hamiltonian cycles HC_i since $\deg_{G_i}(a,i)=3$ and two edges of G_i incident to (a,i) are on HC_i, for $i=1, 2$. Let $HP_1((a,1),(b,1))=HC_1-\{((a,1),(b,1))\}$ and $HP_2((b,2),(a,2))=HC_2-\{((a,2),(b,2))\}$. Hence, we can construct a Hamiltonian cycle $\langle (a,1) \xrightarrow{HP_1((a,1),(b,1))} (b,1) \rightarrow (b,2) \xrightarrow{HP_2((b,2),(a,2))} (a,2) \rightarrow (a,1) \rangle$ of $(G\times K_2)-F_n$, as illustrated in Figure 10.

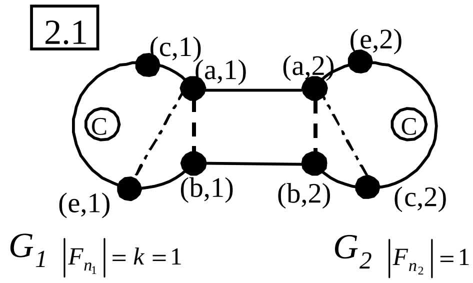


Figure 10 The Hamiltonian cycle of $(G\times K_2)-F_n$.

Case 2.2: $k>1$ and $|F_{n_i}|=k$, $|F_{n_j}|=1$, $1\le i\ne j\le 2$. Without loss of generality, we assume that $|F_{n_1}|=k$, $|F_{n_2}|=1$.

There exists a Hamiltonian cycle HC_1 in $G_1-F_{n_1}$ since G_1 is k-node-Hamiltonian. Let $((a,1),(b,1))$ be an edge in HC_1 such that $(a,2)$ and $(b,2)\notin F$ and $HP_1((a,1),(b,1))=HC_1-\{((a,1),(b,1))\}$. And there is a Hamiltonian path $HP_2((a,2),(b,2))$ in $G_2-F_{n_2}$ since $G_2-F_{n_2}$ is Hamiltonian-connected. Hence, we can construct a Hamiltonian cycle $\langle (a,1) \xrightarrow{HP_1((a,1),(b,1))} (b,1) \rightarrow (b,2) \xrightarrow{HP_2((b,2),(a,2))} (a,2) \rightarrow (a,1) \rangle$ of $(G\times K_2)-F_n$, as illustrated in Figure 11.

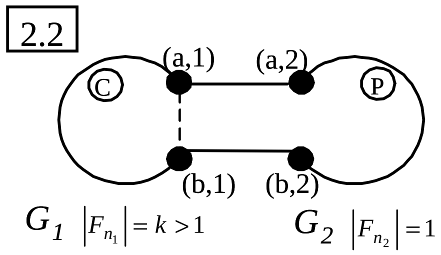


Figure 11 The Hamiltonian cycle of $(G\times K_2)-F_n$.

Case 2.3: $2 \le \left|F_{n_i}\right| \le k\text{-}1$, for $i=1, 2$

Since $\left|F_{n_1}\right| \le k\text{-}1$ and $\left|F_{n_1}\right| \le k\text{-}1$, G_1 and G_2 are Hamiltonian-connected. There exist Hamiltonian paths of G_1 and G_2, respectively. That have the $HP_1((a,1),(b,1))$ and $HP_2((a,2),(b,2))$. Hence there exists a Hamiltonian cycle of $G\times K_2$ $\langle (a,1) \xrightarrow{HP_1((a,1),(b,1))} (b,1) \rightarrow (b,2) \xrightarrow{HP_2((b,2),(a,2))} (a,2) \rightarrow (a,1) \rangle$ of $(G\times K_2)\text{-}F_n$, as illustrated in Figure 12.

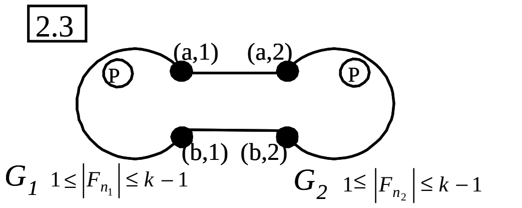


Figure 12 The Hamiltonian cycle of $(G\times K_2)\text{-}F_n$.

According to the above proof, we can obtain that $G\times K_2$ is $(k+1)$-node-Hamiltonian.

In the following, we will show that for all $((s,i),(d,j)) \in V(G\times K_2)$, $1 \le i, j \le 2$, there exists a Hamiltonian path $HP((s,i),(d,j))$ from (s,i) to (d,j) in $(G\times K_2)\text{-}F_n$ for $F_n \subset V(G\times K_2)$ and $|F_n|=k$, with the following cases.

Case 3: $\left|F_{n_1}\right| = k$ or $\left|F_{n_2}\right| = k$. Without loss of generality, we assume that when $\left|F_{n_1}\right| = k$.

Case 3.1: $i=1, j=1$.

Since the $\kappa_h(G_1) = k$, $G_1\text{-}(F_{n_1}\text{-}\{(f,1)\})$ is Hamiltonian-connected for $(f,1) \in F_{n_1}$. Then $G_1\text{-}(F_{n_1}\text{-}\{(f,1)\})$ contains a Hamiltonian path $HP_1((s,1),(d,1)) = \langle (s,1) \rightarrow \cdots \rightarrow (a,1) \rightarrow (f,1) \rightarrow (b,1) \rightarrow \cdots \rightarrow (d,1) \rangle$. Thus $HP_1\text{-}\{(f,1)\}$ forms two paths $P((s,1),(d,1)) = \langle (s,1) \rightarrow \cdots \rightarrow (a,1) \rangle$ and $Q((b,1),(d,1)) = \langle (b,1) \rightarrow \cdots \rightarrow (d,1) \rangle$. Since G_2 is Hamiltonian-connected, there exists a Hamiltonian path $HP_2((a,2),(b,2))$. Hence, we can construct a Hamiltonian path $(s,1)$ to $(d,1)$ $\langle (s,1) \xrightarrow{P((s,1),(a,1))} (a,1) \rightarrow (a,2) \xrightarrow{HP_2((a,2),(b,2))} (b,2) \rightarrow (b,1) \xrightarrow{Q((b,1),(d,1))} (d,1) \rangle$ of $(G\times K_2)\text{-}F_n$, as illustrated in Figure 13.

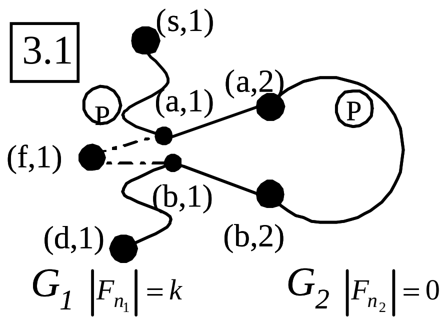


Figure 13 The Hamiltonian path from $(s,1)$ to $(d,1)$ of $(G\times K_2)\text{-}F_n$.

Case 3.2: $i=1, j=2$

Because G_1 is k-node-Hamiltonian, we can find a Hamiltonian cycle HC_1 in $G_1\text{-}F_{n_1}$. If $(a,1) \ne (d,1)$, we can construct a Hamiltonian path from $(s,1)$ to $(d,2)$ $HC_1\text{-}\{((s,1),(a,1))\} \cup \{((a,1),(a,2))\} \cup HP_2((a,2),(d,2))$. On the other hand, if $(a,1)=(d,1)$ then $(b,1) \ne (d,1)$. The similar Hamiltonian path from $(s,1)$ to $(d,2)$ can be constructed as follows: $HC_1\text{-}\{((s,1),(b,1))\} \cup \{((b,1),(b,2))\} \cup HP_2((b,2),(d,2))$ of $(G\times K_2)\text{-}F_n$, as illustrated in Figure 14.

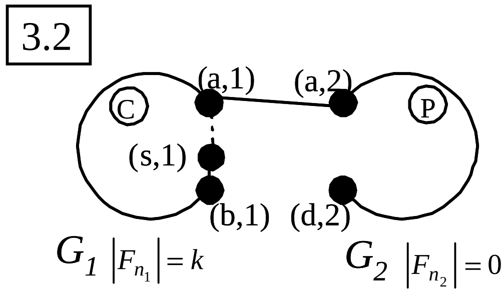


Figure 14 The Hamiltonian path from $(s,1)$ to $(d,2)$ of $(G\times K_2)\text{-}F_n$.

Case 3.3: $i=2, j=2$

Sub-case 3.3.1: $k=1$

When $k=1$, it implies there is a fault in G_1. Since G_1 is 1-node-Hamiltonian, there exists a Hamiltonian cycle in G_1. Since G_2 is Hamiltonian-connected, there is a Hamiltonian path $HP_2((s,2),(d,2))$ in G_2. Let $(a,1) \notin F$ and $(a,2) \ne (s,2)$ and $(a,2) \ne (d,2)$. Since two edges of G_1 incident to $(a,1)$ are on HC_1 and two edges of G_2 incident to $(a,2)$ are on $HP_2((s,2),(d,2))$, there exist $((b,1),(a,1))$ and $((b,2),(a,2))$ are on HC_1 and $HP_2((s,2),(d,2))$, respectively. Therefore, we can construct a Hamiltonian path between $(s,2)$ and $(d,2)$ as $HC_1 \cup HP_2((s,2),(d,2))\text{-}\{((a,1),(b,1)), ((a,2),(b,2))\} \cup \{((a,1),(a,2),(b,1),(b,2))\}$ of

$(G\times K_2)$-F_n, as illustrated in Figure 15.

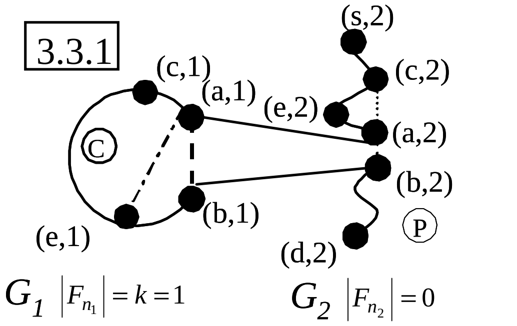


Figure 15 The Hamiltonian path from $(s,2)$ to $(d,2)$ of $(G\times K_2)$-F_n.

Sub-case 3.3.2: $k=2$

Sub-case 3.3.2(1): $(s,1) \notin F_n$ or $(d,1) \notin F_n$. Without loss of generality, we assume that $(s,1)\notin F_n$.

Since G_1 is 2-node-Hamiltonian and the $\kappa_h(G)=2$, there exists a Hamiltonian cycle HC_1 in G_1- F_{n_1} and G_2-$\{(s,2)\}$ is Hamiltonian-connected. Let $((a,1),(s,1))$ and $((b,1),(s,1))$ be two edges of HC_1. One of $(a,1)$and $(b,1)$,say $(b,1)$, is not $(d,1)$. Let $HP_2((b,2),(d,2))$ be a Hamiltonian path of G_2-$(s,2)$. Thus we can construct a Hamiltonian path from $(s,2)$ to $(d,2)$ as $\langle (s,2) \rightarrow (s,1) \xrightarrow{\text{HP}_1\ ((s,1),(b,1))} (b,1) \rightarrow (b,2) \xrightarrow{\text{HP}_2\ ((b,2),(d,2))} (d,2) \rangle$ of $(G\times K_2)$-F_n, as illustrated in Figure 16.

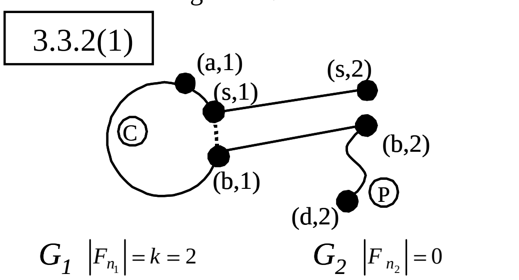


Figure 16 The Hamiltonian path from $(s,2)$ to $(d,2)$ of $(G\times K_2)$-F_n.

Sub case 3.3.2(2): $(s,1)\in F_n$ and $(d,1)\in F_n$.

Since G_2 is Hamiltonian-connected, there exists a Hamiltonian path from $(s,2)$ to $(d,2)$. There exists $((s,2),(a,2))$ which is an edge in G_2, but $((s,2),(a,2))$ isn't the edge on $HP_2((s,2),(d,2))$. Hence the node $(a,2)$ has other three edges and $HP_2((s,2),(d,2))$ must pass through two of these three edges. Because $|F_{n_1}| = k$, we can find a HC_1 of G_1- F_{n_1}. Since $F_n=\{(s,1),(d,1)\}$, $((s,1),(d,1))$ is an edge of HC_1. Thus HC_1 must pass through two of the three edges incident to $(a,1)$. Consequently, there exist edges $((a,1),(b,1))$ and $((a,2),(b,2))$ on HC_1 and HP_2 $((s,2),(d,2))$, respectively. Hence, we can construct a Hamiltonian path from $(s,2)$ to $(d,2)$ $HC_1 \cup HP_2$ $((s,2),(d,2))$-$\{((a,1),(b,1)),((a,2),(b,2))\} \cup \{((a,1),(a,2)),((b,1),(b,2))\}$of $(G\times K_2)$-F_n, as illustrated in Figure 17.

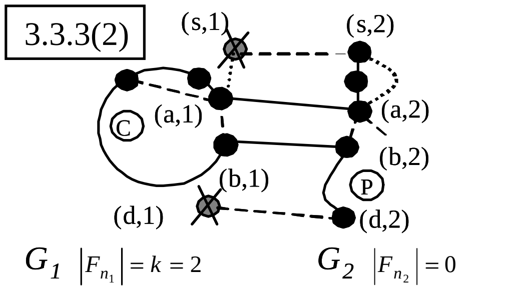


Figure 17 The Hamiltonian path from $(s,2)$ to $(d,2)$ of $(G\times K_2)$-F_n.

Sub case 3.3.3: $k\geq 3$

Sub case 3.3.3(1): $(s,1) \notin F_n$ or $(d,1) \notin F_n$. Without loss of generality, we assume that $(s,1)\notin F_n$.

Since G_1 is k-node-Hamiltonian and the $\kappa_h(G)=k$, for $k\geq 3$, there exists a Hamiltonian cycle HC_1 in G_1- F_{n_1} and G_2-$\{(s,2)\}$ is Hamiltonian-connected. Let $((a,1),(s,1))$ and $((b,1),(s,1))$ be two edges of HC_1. One of $(a,1)$ and $(b,1)$, say $(b,1)$ is not $(d,1)$. Let HP_2 $((s,2),(d,2))$ be a Hamiltonian path of G_2-$(s,2)$. Thus we can construct a Hamiltonian path from $(s,2)$ to $(d,2)$ as $\langle (s,2) \rightarrow (s,1) \xrightarrow{\text{HP}_1\ ((s,1),(b,1))} (b,1) \rightarrow (b,2) \xrightarrow{\text{HP}_2\ ((b,2),(d,2))} (d,2) \rangle$ of $(G\times K_2)$-F_n, as illustrated in Figure 18.

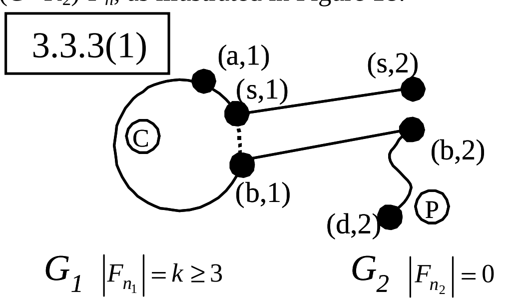


Figure 18 The Hamiltonian path from $(s,2)$ to $(d,2)$ of $(G\times K_2)$-F_n.

Sub case 3.3.3(2): $(s,1)\in F_n$ and $(d,1)\in F_n$.

Since $\deg_{G_2}(s,2)=k+2$, there exists a neighbor (a,i) of (s,i) such that $(a,i)\notin F_n$ for $i=1$, 2. Let HC_1 be a Hamiltonian cycle of G_1-F_1. Let $((b,1),(a,1))$ be an edge on HC_1 and $HP_1((a,1),(b,1))$. Since $\kappa_h(G_2)\geq 3$, there exists a Hamiltonian path $HP_2((b,2),(d,2))$ in G_2-$\{(s,2)(a,2)\}$. Hence, we can construct a Hamiltonian cycle $\langle (s,2) \rightarrow (a,2) \rightarrow (a,1) \xrightarrow{HP_1((a,1),(b,1))} (b,1) \rightarrow (b,2) \xrightarrow{HP_2((b,2),(d,2))} (d,2) \rangle$ of $(G\times K_2)$-F_n, as illustrated in Figure 19.

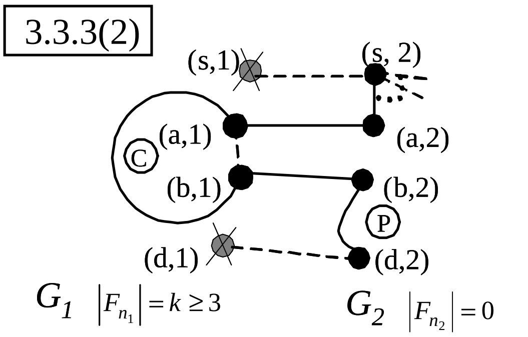


Figure 19 The Hamiltonian path from $(s,2)$ to $(d,2)$ of $(G\times K_2)$-F_n.

Case 4: $1\leq |F_{n_1}| \leq k-1$ and $1\leq |F_{n_2}| \leq k-1$

Sub case 4.1: $i=j$, without loss of generality, we assume $i=1$, $j=1$.

Sub case 4.1.1: $|F_{n_1}| = k-1$, $|F_{n_2}|=1$

Since the faults are not all included in G_1 or G_2, G_1- F_{n_1} and G_2- F_{n_2} are both Hamiltonian-connected. Let $HP_1((s,1),(d,1))$ be a Hamiltonian path of G- F_{n_1}. Let $(a,1)$ and $(a,2)$ be fault-free nodes of G_1 and G_2, for $(a,1)\neq(s,1)$ or $(d,1)$, respectively. There are two edges $((x,1),(a,1))$ and $((a,1),(y,1))$ on $HP_1((s,1),(d,1))$. There exist one of $(x,1)$ and $(y,1)$, say $(x,1)$, such that $(x,2)$ are fault-free since $|F_{n_2}|=1$. Furthermore, there exists a Hamiltonian path $HP_2((x,2),(a,2))$ in G_2-$|F_{n_2}|$. Hence, we can construct a Hamiltonian path from (s,1) to (d,1) $HP_1((s,1),(d,1))$-$\{((a,1),(x,1))\} \cup \{((a,1),(a,2)),((x,1),(x,2))\} \cup HP_2((x,2),(a,2))$ of $(G\times K_2)$-F_n, as illustrated in Figure 20.

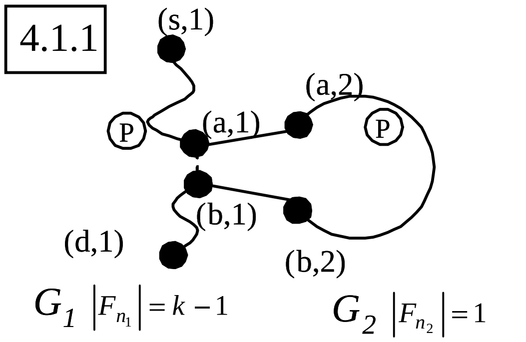


Figure 20 The Hamiltonian path from $(s,1)$ to $(d,1)$ of $(G\times K_2)$-F_n.

Sub case 4.1.2: $|F_{n_1}|=k-2$ and $|F_{n_2}|=2$.

Sub case 4.1.2(1): $(s,2)\notin F_n$ or $(d,2)\notin F_n$. Without loss of generality, we assume that $(s,2)\notin F_n$.

Let $(b,1)\in(V(G_1)$- $F_{n_1})$ and $(b,2)\in(V(G_2)$- $F_{n_2})$ such that $(b,1)\neq(s,1)$ and $(b,1)\neq(d,1)$. Let $HP_1((b,1),(d,1))$ and $HP_2((s,2),(b,2))$ be the Hamiltonian paths of G_1-F_1-$\{(s,1)\}$ and G_2-F_2, respectively. Hence, we can construct a Hamiltonian cycle $\langle (s,1) \rightarrow (s,2) \xrightarrow{HP_2((s,2),(b,2))} (b,2) \rightarrow (b,1) \xrightarrow{HP_1((b,1),(d,1))} (d,1) \rangle$ of $(G\times K_2)$-F_n, as illustrated in Figure 21.

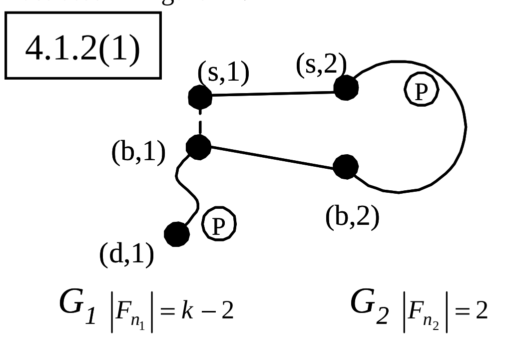


Figure 21 The Hamiltonian path from $(s,1)$ to $(d,1)$ of $(G\times K_2)$-F_n.

Sub case 4.1.2(2): $(s,1)\in F_n$ and $(d,1)\in F_n$.

Since G_1 is Hamiltonian-connected, there exists a Hamiltonian path $HP_1((s,1),(d,1))$. Let $((a,1),(b,1))$ be an edge of $HP_1((s,1),(d,1))$ such that $\{(a,1),(b,1)\} \cap \{(s,1),(d,1)\}=\emptyset$. Since $K_1(G_2)>2$, there exists a Hamiltonian path $HP_2((a,2),(b,2))$ in G_2-F_2. Hence, we can construct a Hamiltonian path from $(s,1)$ to $(d,1)$ $HP_1((s,1),(d,1)) \cup HP_2((a,2),(b,2))$-$\{(a,1),(b,1)\}$

$\cup$ {((*a*,*1*),(*a*,*2*)),((*b*,*1*),(*b*,*2*))} of $(G\times K_2)$-F_n, as illustrated in Figure 22.

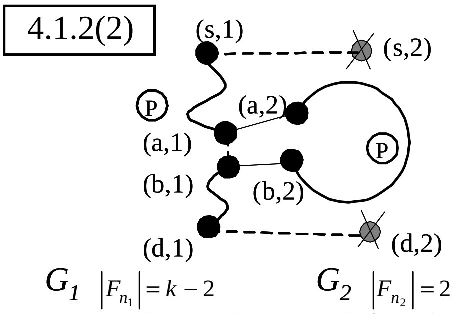


Figure 22 The Hamiltonian path from (*s*,*1*) to (*d*,*1*) of $(G\times K_2)$-F_n.

Sub case 4.1.3: $1 < |F_{n_1}| \le k\text{-}3$ and $3 \le |F_{n_2}| < k$.

Since G_i are *k+2*-regular, there exists a fault-free neighbor (*a*,*i*) of (*s*,*i*) for $(a,i) \neq (d,i)$, *i*=*1*, *2* .Let (*b*,*i*) a be fault-free node such that $(b,i) \notin \{(s,i),(d,i),(a,i)\}$ for *i*= *1*, *2*. Since $|F_{n_1}| \le k\text{-}3$, G_1- F_{n_1} -{(*s*,*1*),(*a*,*1*)} is Hamiltonian-connected. There exist Hamiltonian paths HP_1 ((*b*,*1*),(*d*,*1*)) and HP_2((*a*,*2*),(*b*,*2*)) in G_1- F_{n_1} -{(*s*,*1*),(*a*,*1*)} and G_2-F_2 , respectively. Hence, we can construct a Hamiltonian path $\langle (s,1) \to (a,1) \to (a,2) \xrightarrow{HP_2\,((a,2),(b,2))} (b,2) \to (b,1) \xrightarrow{HP_1\,((b,1),(d,1))} (d,1) \rangle$ of $(G\times K_2)$-F_n, as illustrated in Figure 23.

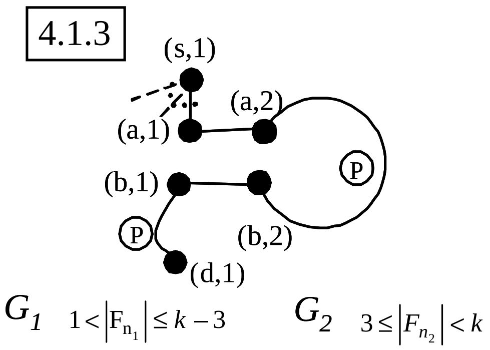


Figure 23 The Hamiltonian path from (*s*,*1*) to (*d*,*1*) of $(G\times K_2)$-F_n.

Sub case 4.2: $i \neq j$, without loss the generality, we assume *i*=*1*, *j*=*2*.

There exists a node $(a,1) \in V(G_1)$ in which $(a,1) \neq (s,1)$, $(a,1) \neq (d,1)$ and (*a*,*1*), (*a*,*2*) and ((*a*,*1*),(*a*,*2*)) are fault-free. Since $|F_{n_1}| < k$ and $|F_{n_2}| < k$, G_1- F_{n_1} and G_2- F_{n_2} are Hamiltonian-connected. Let HP_1((*s*,*1*),(*a*,*1*)) and HP_2 ((*a*,*2*),(*d*,*2*)) denote the Hamiltonian paths of G_1 and G_2, respectively. Therefore, $G\times K_2$-F_n has a Hamiltonian path from (*s*,*1*) to (*d*,*2*) $\langle (s,1) \xrightarrow{HP_1\,((s,1),(a,1))} (a,1) \to (a,2) \xrightarrow{HP_2\,((a,2),(d,2))} (d,2) \rangle$ of $(G\times K_2)$-F_n, as illustrated in Figure 24.

This theorem is proved. □

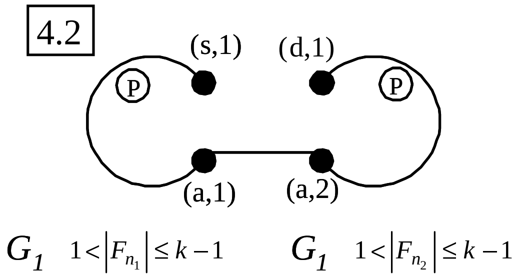
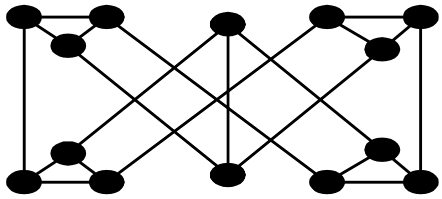


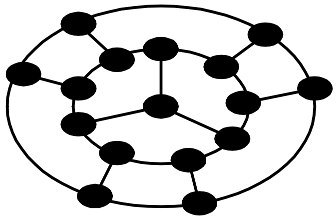
Figure 24 The Hamiltonian path from (*s*,*1*) to (*d*,*2*) of $(G\times K_2)$-F_n.

Applying previous three theorems, we can construct a sequence of graphs which are (*k+2*)-regular, *k*-Hamiltonian (*k*-edge-Hamiltonian, *k*-node-Hamiltonian) and $\sigma_h(G) = k$ ($\lambda_h(G) = k$, $\kappa_h(G) = k$).

The complete graph K_n is a (*n-1*)-regular, (*n-3*)-Hamiltonian graph and $\sigma_h(K_n) = n-3$. Applying Theorem 1 to K_n, we can obtain a series graphs *G* which are (*k+2*)-regular, *k*-Hamiltonian and $\sigma_h(G) = k$. The graph G_1 in Figure 25 is 1-edge-Hamiltonian and $\lambda_h(G_1) = 1$. Applying Theorem 2 to the G_1, we can construct a family of (*k+2*)-regular and *k*-edge-Hamiltonian graphs whose edge-Hamiltonian-connectivity is *k*. The graph G_2 in Figure 25 is 1-node-Hamiltonian and $\kappa_h(G_2) =$ 1. Applying Theorem 3 to G_2, we can construct a family of (*k+2*)-regular and *k*-node-Hamiltonian graphs whose node-Hamiltonian-connectivity is *k*.



G_1



G_2

Figure 25 The 1-node-Hamitlonian graph G_1 and 1-node-Hamiltonian graph G_2

4. Conclusions

In this paper we introduce the concepts of Hamiltonian-connectivity, edge-Hamiltonian-connectivity and node-Hamiltonian-connectivity. Moreover, we present a construction scheme for graphs containing fault tolerance for Hamiltonian and Hamiltonian-connected properties.

In the future, we hope to present more construction schemes for the graphs with these good properties. The relationship about these properties and other Hamiltonian properties (e.g. Hamiltonian-laceability) is worthy to studied.

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