Fault Free Rings and Linear Arraies Embedded in the Faulty (n, k)-star Networks

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Abstract

In this paper, we consider the embedding of hamimilton cycle and the hamiltonian paths between any two vertices of the (n, k)-star graph $S_{n,k}$. Assume that $F \subset V(S_{n,k}) \cup E(S_{n,k})$. For $n-k \geq 2$, we prove that $S_{n,k} - F$ is hamiltonian if $|F| \leq n-3$ and $S_{n,k} - F$ is hamiltonian connected if $|F| \leq n-4$. For $n-k=1$, $S_{n,n-1}$ is isomorphic to the star graph S_n and it is known that S_n is hamiltonian if and only if $n > 2$ and S_n is hamiltonian connected if and only if $n = 2$. Moreover, all the bounds are tight.

Keywords: hamiltonian cycle, hamiltonian connected, (n, k)-star graph.

1 Introduction

The architecture of an interconnection network is usually represented by a graph. There are a lot of mutually conflicting requirements in designing the topology of interconnection networks. It is almost impossible to design a network which is optimum from all aspects. One has to design a suitable network depending on the requirements of its properties. The hamiltonian property is one of the major requirements in designing the topology of network. Fault tolerance is also desirable in the massive parallel systems ,which have relative high probability of failure.

In this paper, a network is represented as a loopless undirected graph. For the graph definition and notation we follow [2]. $G = (V, E)$ is a graph if V is a finite set and E

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is a subset of $\{(u,v) \mid (u,v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices u and v are *adjacent* if $(u,v) \in E$. A *path* is represented by $\langle v_0, v_1, v_2, \cdots, v_k \rangle$. The *length* of a path P is the number of edges in P. We also write the path $\langle v_0, v_1, v_2, \cdots, v_k \rangle$ as $\langle v_0, P_1, v_i, v_{i+1}, \cdots, v_j, P_2, v_t, \cdots, v_k \rangle$, where P_1 is the path $\langle v_0, v_1, \cdots, v_i \rangle$ and P_2 is the path $\langle v_j, v_{j+1}, \cdots, v_t \rangle$. Hence, it is possible to write a path as $\langle v_0, v_1, P, v_1, v_2, \ldots, v_k \rangle$ if the length of P is 0. We use $d(u,v)$ to denote the distance between u and v, i.e., the length of the shortest path joining u and v. A path is a *hamiltonian path* if its vertices are distinct and span V. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last vertex. A cycle is a *hamiltonian cycle* if it traverses every vertex of G exactly once. A graph is *hamiltonian* if it has a hamiltonian cycle.

Since vertex faults and edge faults may happen when a network is used, they are practically meaningful to the faulty networks we consider. The vertex fault-tolerant hamiltonicity and the edge fault-tolerant hamiltonicity, proposed by Hsieh, Chen, and Ho [4], measure the performance of the hamiltonian property in the faulty networks. A hamiltonian graph G is *k-vertex-fault hamiltonian* if $G - F$ remains hamiltonian for every $F \subset V(G)$ with $|F| \leq k$. The *vertex fault-tolerant hamiltonicity*, $\mathcal{H}_v(G)$, is defined to be the maximum integer k such that G is k-vertex-fault hamiltonian if G is hamiltonian and undefined if otherwise. Obviously, $\mathcal{H}_v(G) \leq \delta(G) - 2$ where $\delta(G) = \min\{\deg(v) \mid v \in V(G)\}$ if $\mathcal{H}_v(G)$ is defined. Similarly, a hamiltonian graph G is *k-edge-fault hamiltonian* if $G - F$ remains hamiltonian for every $F \subset E(G)$ with $|F| \leq k$. The *edge fault-tolerant hamiltonicity*, $\mathcal{H}_e(G)$, is defined to be the maximum integer k such that G is k-edge-fault hamiltonian if G is hamiltonian and undefined if otherwise. Again, it is easy to see that $\mathcal{H}_e(G) \leq \delta(G) - 2$ if $\mathcal{H}_e(G)$ is defined. Huang et al. [5] define a more general parameter, fault-tolerant hamiltonicity. A hamiltonian graph G is *k-fault hamiltonian* if $G - F$ remains hamiltonian for every $F \subset V(G) \cup E(G)$ with $|F| \leq k$. The *fault-tolerant hamiltonicity*, $\mathcal{H}_f(G)$, is defined to be the maximum integer k such that G is k-fault hamiltonian if G is hamiltonian and undefined if otherwise. Clearly, $\mathcal{H}_f(G) \leq \delta(G) - 2$ if $\mathcal{H}_f(G)$ is defined. Huang et al. [5] also introduced the term, fault-tolerant hamiltonian connected. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two vertices of G. All hamiltonian connected graphs except K_1 and K_2 are hamiltonian. A graph G is *k-fault hamiltonian connected* if $G - F$ remains hamiltonian connected for every $F \subset V(G) \cup E(G)$ with $|F| \leq k$. The *fault-tolerant hamiltonian connectivity*, $\mathcal{H}_f^{\kappa}(G)$, is defined to be the maximum integer k such that G is k-fault hamiltonian connected if G is hamiltonian connected and undefined if otherwise. It can be checked that $\mathcal{H}_f^{\kappa}(G) \leq \delta(G) - 3$ only if $\mathcal{H}_f^{\kappa}(G)$ is defined and $|V(G)| \geq 4$.

The n-star graph [1] is an attractive alternative to the n-cube. It has many significant advantages over the n-cube, such as a lower degree and a smaller diameter. However, a major practical difficulty with the n-star graph is the restriction on the number of vertices: $n!$ for an n-star graph. In order to remedy this drawback, the (n, k)-star graph is proposed [3] as an attractive alternative to the n-star graph.

Throughout this paper, we assume that n and k are positive integers with $n > k$. We use $\langle n \rangle$ to denote the set $\{1, 2, \ldots, n\}$. The (n, k)-star graph, denoted by $S_{n,k}$, is a graph with the vertex set $V(S_{n,k}) = \{u_1 u_2 \ldots u_k \mid u_i \in \langle n \rangle \text{ and } u_i \neq u_j \text{ for } i \neq j\}$. The adjacency is defined as follows: $u_1 u_2 \ldots u_i \ldots u_k$ is adjacency to (1) $u_i u_2 \ldots u_1 \ldots u_k$ through an edge of dimension i, where $2 \leq i \leq k$ (swap u_i with u_1), and (2) $x u_2 \ldots u_k$ through dimension 1, where $x \in \langle n \rangle - \{u_i \mid 1 \leq i \leq k\}$. The $(4, 2)$-star graph is shown in Figure 1. The edges of type (1) are referred to an *i-edges*, and the edges of type (2) are referred to as a *1-edges*. By definition, $S(n, k)$ is an $(n - 1)$-regular graph with $n!/(n - k)!$ vertices. Moreover, it is vertex transitive. The $(n, n - 1)$-star graph is isomorphic to the n-star graph S_n, and the $(n, 1)$-star graph is isomorphic to the complete graph with n vertices, K_n.

In this paper, we discuss the fault hamiltonian property and the fault hamiltonian connected property of the (n, k)-star graphs. In the following section, we discuss some properties of complete graphs. In section 3, we discuss some properties of the (n, k)-star graphs. In the final section, we prove that (1) $\mathcal{H}_f(S_{n,k}) = n - 3$ and $\mathcal{H}_f^{\kappa}(S_{n,k}) = n - 4$ if $n - k \geq 2$; (2) $\mathcal{H}_f(S_{2,1})$ is undefined and $\mathcal{H}_f^{\kappa}(S_{2,1}) = 0$; and (3) $\mathcal{H}_f(S_{n,n-1}) = 0$ and $\mathcal{H}_f^{\kappa}(S_{n,n-1})$ is undefined if $n > 2$.

2 Some properties of complete graphs

Let $G = (V, E)$ be a graph. We use $\bar{E}$ to denote the edge set in the complement of G. The following Theorem is proved by Ore [7].

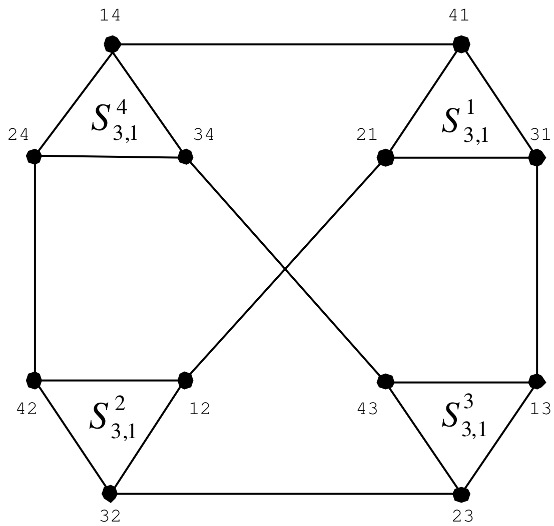

Figure 1: $(4, 2)$-star graph

Theorem 1 *[7] Assume that $G = (V, E)$ is a graph with n vertices with $n > 3$. Then G is hamiltonian if $|\bar{E}| \leq n - 3$, and hamiltonian connected if $|\bar{E}| \leq n - 4$.*

By Theorem 1, there exists a hamiltonian path joining any two different vertices of an n-vertex graph $G = (V, E)$ with $n \geq 4$ and $|\bar{E}| \leq n - 4$. However, we have the following solid result.

Theorem 2 *Assume that $G = (V, E)$ is a graph with $V = \langle n \rangle$, $n \geq 4$, and $|\bar{E}| \leq n - 4$. Then, there are two hamiltonian paths of G joining any two different vertices i and j in V, say $P_1 = \langle i = i_1, i_2, \ldots, i_{n-1}, i_n = j \rangle$ and $P_2 = \langle i = i'_1, i'_2, \ldots, i'_{n-1}, i'_n = j \rangle$, such that $i_2 \neq i'_2$ and $i_{n-1} \neq i'_{n-1}$.*

Proof. We prove this theorem by induction. It is easy to check that the theorem is true for $n = 4$. Assume that the theorem holds for some integer m with $n > m \geq 4$. Let i and j be any two vertices of G. We want to find two hamiltonian paths of G joining i and j, $P_1 = \langle i = i_1, i_2, \ldots, i_{n-1}, i_n = j \rangle$ and $P_2 = \langle i = i'_1, i'_2, \ldots, i'_{n-1}, i'_n = j \rangle$, such that $i_2 \neq i'_2$ and $i_{n-1} \neq i'_{n-1}$. Let X be a subset of $\langle n \rangle$. We use G^X to denote the subgraph of G induced by X and $\bar{E}^X$ denote the set $\{(i, j) \mid (i, j) \in \bar{E}, i, j \in X\}$.

By the symmetric property of the complete graph, we may assume that $|\bar{E}^{\langle n-1 \rangle}| = n - 4$. Hence, $(n, l) \in E$ for any $l \in \langle n - 1 \rangle$.

Suppose that $i = n$ or $j = n$. Without loss of generality, we assume that $i = n$. Since $|\bar{E}^{\langle n-1\rangle}| = n - 4$, by Theorem 1 there is a hamiltonian cycle of $G^{\langle n-1\rangle}$, say $\langle j = a_1, a_2, a_3, \ldots, a_{n-1}, a_1 = j\rangle$. Then $\langle i, a_2, a_3 \ldots, a_{n-1}, a_1 = j\rangle$ and $\langle i, a_{n-1}, a_{n-2}, \ldots, a_2, a_1 = j\rangle$ form two hamiltonian paths of G satisfying our requirement.

Now, we consider that $i \neq n$ and $j \neq n$. Choose any edge e in $\bar{E}^{\langle n-1\rangle}$. By induction, there are two hamiltonian paths of $G^{\langle n-1\rangle} + e$ joining i and j, say $P_1' = \langle i = a_1^1, a_2^1, \ldots, a_{n-2}^1, a_{n-1}^1 = j\rangle$ and $P_2' = \langle i = a_1^2, a_2^2, \ldots, a_{n-2}^2, a_{n-1}^2 = j\rangle$, such that $a_2^1 \neq a_2^2$ and $a_{n-2}^1 \neq a_{n-2}^2$. For $l = 1, 2$, we set P_l as $\langle i = a_1^l, a_2^l, \ldots, a_t^l, n, a_{t+1}^l, \ldots, a_{n-2}^l, a_{n-1}^l = j\rangle$ where $t = 2$ if $(a_p^l, a_{p+1}^l) \neq e$ for all $1 \leq p \leq n-1$, or t is the index p such that $(a_p^l, a_{p+1}^l) = e$. Obviously, P_1 and P_2 form two hamiltonian paths of G satisfying our requirement.

Hence, the theorem is proved. $\square$

Lemma 1 *Assume that $n \geq 4$. Then K_n is $(n-3)$-fault hamiltonian and $(n-4)$-fault hamiltonian connected.*

The following theorem was proved by Hung et al. [6].

Theorem 3 *[6] Let $K_n = (V, E)$ be the complete graph with n vertices. Let $F \subset (V \cup E)$ be a faulty set with $|F| \leq n - 2$. Then there exists a vertex set $V' \subseteq V(K_n) - F$ with $|V'| = n - |F|$ such that there exists a hamiltonian path of $K_n - F$ joining every pair of vertices in V'.*

3 Basic properties of the (n, k)-star graphs

Let $\mathbf{u} = u_1 u_2 \ldots u_k$ be any vertex of the (n, k)-star graph. We say u_i is the ith coordinate of $\mathbf{u}$, denoted by $(\mathbf{u})_i$, for $1 \leq i \leq k$. Let $\mathbf{v}$ be a neighbor of $\mathbf{u}$. We say that $\mathbf{v}$ is an *i-neighbor* of $\mathbf{u}$ if $u_i \neq v_i$. By the definition of $S_{n,k}$, there is exactly one i-neighbor of $\mathbf{u}$ for $2 \leq i \leq k$ and there are $(n-k)$ 1-neighbors of $\mathbf{u}$. For this reason, we use $i(\mathbf{u})$ to denote the unique i-neighbor of $\mathbf{u}$ if $i \neq 1$. In particular, $(k(\mathbf{u}))_k = (\mathbf{u})_1$. For $1 \leq i \leq n$, let $S_{n-1,k-1}^i$ be the subgraph of $S_{n,k}$ induced by those vertices $\mathbf{u}$ with $(\mathbf{u})_k = i$. In [3], Chiang et. al. proved that $S_{n,k}$ can be decomposed into n subgraphs $S_{n-1,k-1}^i$, $1 \leq i \leq n$, such that each subgraph $S_{n-1,k-1}^i$ is isomorphic to $S_{n-1,k-1}$. Thus, the (n, k)-star graph can be constructed recursively.

Lemma 2 *Let $n > k > 1$ and $\mathbf{u}$ and $\mathbf{v}$ be two vertices in $S^l_{n-1,k-1}$ with $d(\mathbf{u},\mathbf{v}) \leq 2$ for some $1 \leq l \leq n$. Then $(k(\mathbf{u}))_k \neq (k(\mathbf{v}))_k$.*

Proof. Let $\mathbf{u} = u_1u_2\cdots u_k$. Suppose that $d(\mathbf{u},\mathbf{v}) = 1$. Since every edge in $S^l_{n-1,k-1}$ is an i-edge with $1 \leq i < k$, $\mathbf{v}$ is either $i(\mathbf{u})$ for some $2 \leq i < k$ or $xu_2\cdots u_k$ for some $x \in \langle n\rangle - \{u_j \mid 1 \leq j \leq k\}$. Obviously, $(k(\mathbf{u}))_k$ is u_1 and $(k(\mathbf{v}))_k$ is either u_i with $2 \leq i < k$ or x. Hence, $(k(\mathbf{u}))_k \neq (k(\mathbf{v}))_k$.

Suppose that $d(\mathbf{u},\mathbf{v}) = 2$. Then there is a vertex $\mathbf{w} \in V(S^l_{n-1,k-1})$, such that $d(\mathbf{u},\mathbf{w}) = d(\mathbf{v},\mathbf{w}) = 1$. Let $\mathbf{w} = w_1w_2\cdots w_k$.

Assume that (1) $\mathbf{u}$ is $i(\mathbf{w})$ and (2) $\mathbf{v}$ is either $j(\mathbf{w})$ for some $2 \leq j \neq i < k$ or $xw_2\cdots w_k$ for some $x \in \langle n\rangle - \{w_r \mid 1 \leq r \leq k\}$. Thus, $(k(\mathbf{u}))_k = w_i$. Moreover, $(k(\mathbf{v}))_k = w_j$ or $(k(\mathbf{v}))_k = x$ with $2 \leq i \neq j < k$. Hence, $(k(\mathbf{u}))_k \neq (k(\mathbf{v}))_k$.

Assume that (1) $\mathbf{u}$ is $x_1w_2\cdots w_k$ for some $x_1 \in \langle n\rangle - \{w_r \mid 1 \leq r \leq k\}$ and (2) $\mathbf{v}$ is $x_2w_2\cdots w_k$ for some $x_2 \in \langle n\rangle - \{w_r \mid 1 \leq r \leq k\}$ with $x_1 \neq x_2$. Thus, $(k(\mathbf{u}))_k = x_1$ and $(k(\mathbf{v}))_k = x_2$. Hence, $(k(\mathbf{u}))_k \neq (k(\mathbf{v}))_k$.

Thus, the lemma is proved. $\square$

For $1 \leq i \neq j \leq n$, we use $E^{i,j}$ to denote the set of edges between $S^i_{n-1,k-1}$ and $S^j_{n-1,k-1}$. Let $(\mathbf{u},\mathbf{v})$ be any edge in $E^{i,j}$. In the following, we assume that $\mathbf{u} \in S^i_{n-1,k-1}$ and $\mathbf{v} \in S^j_{n-1,k-1}$. Thus, $(\mathbf{u},\mathbf{v}) \in E^{i,j}$ implies $(\mathbf{v},\mathbf{u}) \in E^{j,i}$. However, $(\mathbf{u},\mathbf{v}) \notin E^{j,i}$ if $(\mathbf{u},\mathbf{v}) \in E^{i,j}$. In [3], it is proved that $|E^{i,j}| = \frac{(n-2)!}{(n-k)!}$. Thus, $|E^{i,j}| = 1$ if $k = 2$. The following lemma can be easily obtained from the definition of $S_{n,k}$.

Lemma 3 *Let $(\mathbf{u},\mathbf{v})$ and $(\mathbf{u}',\mathbf{v}')$ be any two distinct edges in $E^{i,j}$. Then, $\{\mathbf{u},\mathbf{v}\} \cap \{\mathbf{u}',\mathbf{v}'\} = \emptyset$.*

Let F be a faulty set of $S_{n,k}$. An edge $(\mathbf{u},\mathbf{v})$ is *F-fault* if $(\mathbf{u},\mathbf{v}) \in F$, $\mathbf{u} \in F$, or $\mathbf{v} \in F$; and $(\mathbf{u},\mathbf{v})$ is *F-fault free* if $(\mathbf{u},\mathbf{v})$ is not F-fault. Let $H = (V', E')$ be a subgraph of $S_{n,k}$. We use $F(H)$ to denote the set $(V' \cup E') \cap F$. We associate $S_{n,k}$ with the complete graph K_n with vertex set $\langle n\rangle$ such that vertex l of K_n is associated with $S^l_{n-1,k-1}$ for every $1 \leq l \leq n$. We define a faulty edge set $R(F)$ of K_n as $(i,j) \in R(F)$ if some edge of $E^{i,j}$ is F-fault. Obviously, $|R(F)| \leq |F|$. Assume that I is any subset of $\langle n\rangle$. We use $S^I_{n-1,k-1}$ to denote the subgraph of $S_{n,k}$ induced by $\cup_{i\in I} V(S^i_{n-1,k-1})$. Similarly, we use K^I_n to denote the subgraph of K_n induced by I.

Lemma 4 *Suppose that $k \geq 2$, and $(n-k) \geq 2$. Let $I \subseteq \langle n \rangle$ with $|I| = m \geq 2$ and let $F \subset V(S_{n,k}) \cup E(S_{n,k})$ with $S^i_{n-1,k-1} - F$ being hamiltonian connected for all $i \in I$. Let $\mathbf{u}$ and $\mathbf{v}$ be any two vertices of $S^I_{n-1,k-1}$ such that (1) $(\mathbf{u})_k \neq (\mathbf{v})_k$, (2) there exists a hamiltonian path $P = \langle (\mathbf{u})_k = i_1, i_2, \ldots, i_m = (\mathbf{v})_k \rangle$ of $K^I_n - R(F)$, and (3) $(k(\mathbf{u}))_k \neq i_2$ and $(k(\mathbf{v}))_k \neq i_{m-1}$ if $k = 2$. Then there exists a hamiltonian path of $S^I_{n-1,k-1} - F$ joining $\mathbf{u}$ to $\mathbf{v}$.*

Proof. Let $\mathbf{u}^1 = \mathbf{u}$ and $\mathbf{v}^m = \mathbf{v}$. Suppose that we can choose two different vertices $\mathbf{u}^l$ and $\mathbf{v}^l$ in $S^{i_l}_{n-1,k-1}$ for every $i_l \in I$ such that $(\mathbf{v}^l, \mathbf{u}^{l+1}) \in E^{i_l, i_{l+1}}$. Since $(i_l, i_{l+1}) \notin R(F)$, $(\mathbf{v}^l, \mathbf{u}^{l+1})$ is F-fault free. Since $S^{i_l}_{n-1,k-1} - F$ is hamiltonian connected, there exists a hamiltonian path P_l of $S^{i_l}_{n-1,k-1} - F$ joining $\mathbf{u}^l$ and $\mathbf{v}^l$ for all $1 \leq l \leq m$. Then $\langle \mathbf{u} = \mathbf{u}^1, P_1, \mathbf{v}^1, \mathbf{u}^2, P_2, \mathbf{v}^2, \ldots, \mathbf{u}^m, P_m, \mathbf{v}^m = \mathbf{v} \rangle$ forms a hamiltonian path of $S^I_{n-1,k-1} - F$ joining $\mathbf{u}$ to $\mathbf{v}$. Thus, the lemma is proved once such a choice is achievable.

Suppose that $k \geq 3$. Since $|E^{i,j}| \geq (n-2) \geq 3$ for any i and j in I, such a choice is easily achievable. Suppose that $k = 2$. We can choose $\mathbf{u}^l$ and $\mathbf{v}^l$ in $S^{i_l}_{n-1,k-1}$ for every $i_l \in I$ as the only edge $(\mathbf{v}^l, \mathbf{u}^{l+1}) \in E^{i_l, i_{l+1}}$. Since $(k(\mathbf{u}^l))_k = (\mathbf{v}^{l-1})_k \neq (k(\mathbf{v}^l))_k = (\mathbf{u}^{l+1})_k$, $\mathbf{u}^l \neq \mathbf{v}^l$ for every $1 < l < m$. The conditions $(k(\mathbf{u}))_k \neq i_2$ and $(k(\mathbf{v}))_k \neq i_{m-1}$ imply $\mathbf{u} = \mathbf{u}^1 \neq \mathbf{v}^1$ and $\mathbf{v} = \mathbf{v}^m \neq \mathbf{u}^m$. Hence, the lemma is proved. □

4 Hamiltonian properties of the (n,k)-star graphs

Lemma 5 *$S_{4,2}$ is 1-fault hamiltonian and hamiltonian connected.*

Proof. It is easy to check it is correct.

Lemma 6 *Suppose that, for some $k \geq 2$, $n \geq 5$, and $n-k \geq 2$, $S_{n-1,k-1}$ is $(n-4)$-fault hamiltonian and $(n-5)$-fault hamiltonian connected. Then $S_{n,k}$ is $(n-3)$-fault hamiltonian.*

Proof. Assume that F is any faulty set of $S_{n,k}$ with $|F| \leq n-3$. Without loss of generality, we assume that $|F(S^1_{n-1,k-1})| \geq |F(S^2_{n-1,k-1})| \geq \cdots \geq |F(S^n_{n-1,k-1})|$.

Case 1: $|F(S^1_{n-1,k-1})| \le n-5$. By the assumption of this lemma, $S^i_{n-1,k-1} - F$ is hamiltonian connected for every $i \in \langle n \rangle$. Since $|R(F)| \le |F| \le n-3$, by Theorem 1 $K_n - R(F)$ is Hamiltonian. Let $C = \langle t_1, t_2, \ldots, t_n, t_1 \rangle$ be a hamiltonian cycle of $K_n - R(F)$. Thus, all edges in E^{t_1,t_n} are F-fault free. We choose any edge $(\mathbf{u}, \mathbf{v})$ in E^{t_1,t_n}. Obviously, $\langle t_1, t_2, \ldots, t_n \rangle$ is a hamiltonian path of $K_n - R(F)$, $(k(\mathbf{u}))_k = (\mathbf{v})_k = t_n$, and $(k(\mathbf{v}))_k = (\mathbf{u})_k = t_1$. By Lemma 4, there exists a hamiltonian path P_1 of $S_{n,k} - F$ joining $\mathbf{u}$ to $\mathbf{v}$. Thus, $\langle \mathbf{u}, P_1, \mathbf{v}, \mathbf{u} \rangle$ forms a hamiltonian cycle of $S_{n,k} - F$.

Case 2: $|F(S^1_{n-1,k-1})| = n-4$. By the assumption of this lemma, $S^1_{n-1,k-1} - F$ is hamiltonian. Suppose that $n \ge 6$. Then $S^i_{n-1,k-1} - F$ is hamiltonian connected for every $i \ne 1$. Let C be a hamiltonian cycle of $S^1_{n-1,k-1} - F$. Since the length of C is at least 3, there exists an edge $(\mathbf{u}, \mathbf{v})$ in C such that $(\mathbf{u}, k(\mathbf{u}))$ and $(\mathbf{v}, k(\mathbf{v}))$ are F-fault free. We can write C as $\langle \mathbf{u}, P_1, \mathbf{v}, \mathbf{u} \rangle$. Let $F' = F - F(S^1_{n-1,k-1})$. Then $|F'| \le 1$ and $|R(F')| \le 1$. Then $K_n^{\langle n \rangle - \{1\}} - R(F')$ is hamiltonian connected and $(k(k(\mathbf{u})))_k = (\mathbf{u})_k = (\mathbf{v})_k = 1$. By Lemma 4, there is a hamiltonian path P_2 of $S^{\langle n \rangle - \{1\}}_{n-1,k-1} - F'$ joining $k(\mathbf{u})$ to $k(\mathbf{v})$. Hence, $\langle \mathbf{u}, k(\mathbf{u}), P_2, k(\mathbf{v}), \mathbf{v}, P_1, \mathbf{u} \rangle$ forms a hamiltonian cycle of $S_{n,k} - F$.

Suppose that $n = 5$. Then $|F| = 2$, $|F(S^1_{n-1,k-1})| = |F(S^2_{n-1,k-1})| = 1$, and $|F - F(S^{\{1,2\}}_{n-1,k-1})| = 0$. By the assumption of this lemma, $S^1_{n-1,k-1} - F$ and $S^2_{n-1,k-1} - F$ are hamiltonian, and $S^i_{n-1,k-1}$ is hamiltonian connected if $i \in \{3,4,5\}$. Obviously, there exists an index r in $\{3,4,5\}$ such that some edges $(\mathbf{u}, k(\mathbf{u}))$ in $E^{1,r}$ and some edges $(\mathbf{v}, k(\mathbf{v}))$ in $E^{2,r}$ are F-fault free. Thus, $k(\mathbf{u}) \ne k(\mathbf{v})$. Let $C_1 = \langle \mathbf{u}, \mathbf{w}, P_1, \mathbf{u} \rangle$ be a hamiltonian cycle of $S^1_{n-1,k-1} - F$ and $C_2 = \langle \mathbf{v}, \mathbf{x}, P_2, \mathbf{x}' \rangle$ be a hamiltonian cycle of $S^2_{n-1,k-1}$. Since the cycle C_2 can be traversed forward and backward, we may assume that $(k(\mathbf{w}))_k \ne (k(\mathbf{x}))_k$. Obviously, $\langle (k(\mathbf{w}))_k, (k(\mathbf{x}))_k \rangle$ is a hamiltonian path of $K_n^{\langle n \rangle - \{1,2,r\}}$, $(\mathbf{w})_k = 1$, and $(\mathbf{x})_k = 2$. By Lemma 4, there exists a hamiltonian path P_3 of $S^{\langle n \rangle - \{1,2,r\}}_{n-1,k-1}$ joining $k(\mathbf{w})$ to $k(\mathbf{x})$. By the assumption of this lemma, there exists a hamiltonian path P_4 of $S^r_{n-1,k-1}$ joining $k(\mathbf{v})$ to $k(\mathbf{u})$. Then $\langle \mathbf{u}, P_1, \mathbf{w}, k(\mathbf{w}), P_3, k(\mathbf{x}), \mathbf{x}, P_2, \mathbf{x}', \mathbf{v}, k(\mathbf{v}), P_4, k(\mathbf{u}), \mathbf{u} \rangle$ forms a hamiltonian cycle of $S_{n,k} - F$.

Case 3: $|F(S^1_{n-1,k-1})| = n-3$. Thus, $|F - F(S^1_{n-1,k-1})| = 0$. Choose any element f in $F(S^1_{n-1,k-1})$. By the assumption of this lemma, there exists a hamiltonian cycle of $S^1_{n-1,k-1} - F + \{f\}$. By deleting f from $S^1_{n-1,k-1} - F$, we can find a hamiltonian path of $S^1_{n-1,k-1} - F$ joining $\mathbf{u}$ and $\mathbf{v}$ such that $d(\mathbf{u}, \mathbf{v}) \le 2$, no matter f is a vertex or an edge. By

Lemma 2, $(k(\mathbf{u}))_k \neq (k(\mathbf{v}))_k$. Since $K_n^{\langle n\rangle-\{1\}}$ is hamiltonian connected and $(\mathbf{u})_k = (\mathbf{v})_k = 1$, by Lemma 4 there exists a hamiltonian path P_2 of $S_{n-1,k-1}^{\langle n\rangle-\{1\}}$ joining $k(\mathbf{u})$ to $k(\mathbf{v})$. Thus, $\langle \mathbf{u}, k(\mathbf{u}), P_2, k(\mathbf{v}), \mathbf{v}, P_1, \mathbf{u}\rangle$ forms a hamiltonian cycle of $S_{n,k} - F$.

Hence, the lemma follows. □

Lemma 7 *$S_{n,2}$ is $(n-4)$-fault hamiltonian connected for $n \geq 5$.*

Proof. By definition, $S_{n-1,1}^i$ is isomorphic to K_{n-1} for every $i \in \langle n\rangle$. Moreover, $|E^{i,j}| = 1$ for any two $i, j \in \langle n\rangle$. Assume that F is any faulty set of $S_{n,2}$ with $|F| \leq n-4$. Without loss of generality, we assume that $|F(S_{n-1,1}^1)| \geq |F(S_{n-1,1}^2)| \geq \cdots \geq |F(S_{n-1,1}^n)|$. Let $\mathbf{x}$ and $\mathbf{y}$ be any two arbitrary vertices of $S_{n,2} - F$. We need to construct a hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ and $\mathbf{y}$.

Case 1: $|F(S_{n-1,1}^1)| \leq n-5$. By Lemma 1, $S_{n-1,1}^t - F(S_{n-1,1}^t)$ is hamiltonian connected for any $t \in \langle n\rangle$.

Subcase 1.1: $(\mathbf{x})_k = (\mathbf{y})_k$. Let $F' = F \cup \{(\mathbf{y}, k(\mathbf{y}))\}$. Then $|R(F')| \leq n-3$. By Theorem 1, there exists a hamiltonian cycle C in $K_n - R(F')$, say $C = \langle (\mathbf{x})_k = a_1, a_2, \cdots, a_n, a_1\rangle$. Thus, the only edge $(\mathbf{u}, k(\mathbf{u}))$ in E^{a_1,a_2} and the only edge $(\mathbf{v}, k(\mathbf{v}))$ in E^{a_1,a_n} are F-fault free.

Suppose that $|F(S_{n-1,1}^{a_1})| = 0$. Since $(\mathbf{y}, k(\mathbf{y})) \in F'$, $\mathbf{v} \neq \mathbf{y}$. Obviously, $\langle a_2, a_3, \cdots, a_n\rangle$ is a hamiltonian path of $K_n^{\langle n\rangle-\{a_1\}} - R(F')$ and $(\mathbf{u})_k = (\mathbf{v})_k = a_1$. By Lemma 4, there exists a hamiltonian path P_1 of $S_{n-1,1}^{\langle n\rangle-\{a_1\}}$ joining $k(\mathbf{u})$ to $k(\mathbf{v})$. Since $S_{n-1,1}^{a_1}$ is a complete graph, there exist two paths P_2 and P_3 covering all vertices in $S_{n-1,1}^{a_1}$ such that P_2 joins $\mathbf{x}$ to $\mathbf{u}$ and P_3 joins $\mathbf{v}$ to $\mathbf{y}$. Then, $\langle \mathbf{x}, P_2, \mathbf{u}, k(\mathbf{u}), P_1, k(\mathbf{v}), \mathbf{v}, P_3, \mathbf{y}\rangle$ forms a hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Suppose that $|F(S_{n-1,1}^{a_1})| \geq 1$. We create a new graph H by setting $V(H) = V(S_{n-1,1}^{a_1}) \cup \{n\}$ and $E(H) = E(S_{n-1,1}^{a_1}) \cup \{(\mathbf{w}, n) \mid \mathbf{w} \in V(S_{n-1,1}^{a_1})\}$. Hence, H is a complete graph with n vertices. Then we set $F'' = F(S_{n-1,1}^{a_1}) \cup \{(\mathbf{w}, n) \mid$ the only edge in $E^{a_1,(k(\mathbf{w}))_k}$ is F-fault$\}$. Hence, $|F''| \leq n-4$. By Theorem 1, $H - F''$ is hamiltonian connected. Thus, there exists a hamiltonian path P_1 of $H - F''$ joining $\mathbf{x}$ to $\mathbf{y}$. Since n is an internal vertex of P_1, we can write P_1 as $\langle \mathbf{x} = \mathbf{u}^1, Q_1, \mathbf{u}^s, n = \mathbf{u}^{s+1}, \mathbf{u}^{s+2}, Q_2, \mathbf{u}^n = \mathbf{y}\rangle$. Since $|F(S_{n-1,1}^{a_1})| \geq 1$, $|R(F(S_{n-1,1}^{\langle n\rangle-\{a_1\}}))| \leq |F - F(S_{n-1,1}^{a_1})| \leq n-5$. By Theorem 1,

$K_n^{\langle n\rangle-\{a_1\}} - R(F(S_{n-1,1}^{\langle n\rangle-\{a_1\}}))$ is hamiltonian connected. Obviously, $(\mathbf{u}^s)_k = (\mathbf{u}^{s+2})_k = a_1$. With Lemma 4, there exists a hamiltonian path P_2 of $S_{n-1,1}^{\langle n\rangle-\{a_1\}} - F$ joining $k(\mathbf{u}^s)$ to $k(\mathbf{u}^{s+2})$. Then $\langle \mathbf{x} = \mathbf{u}^1, Q_1, \mathbf{u}^s, k(\mathbf{u}^s), P_2, k(\mathbf{u}^{s+2}), \mathbf{u}^{s+2}, Q_2, \mathbf{u}^n = \mathbf{y}\rangle$ forms a hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Subcase 1.2: $(\mathbf{x})_k \neq (\mathbf{y})_k$. Since $|F| \leq n-4$, $|R(F)| \leq n-4$. By Theorem 2, there are two hamiltonian paths of $K_n - R(F)$ joining $(\mathbf{x})_k$ to $(\mathbf{y})_k$, say $P_1 = \langle(\mathbf{x})_k = l_1, l_2, \ldots, l_n = (\mathbf{y})_k\rangle$ and $P_2 = \langle(\mathbf{x})_k = l'_1, l'_2, \ldots, l'_n = (\mathbf{y})_k\rangle$, such that $l_2 \neq l'_2$ and $l_{n-1} \neq l'_{n-1}$. Suppose that $((k(\mathbf{x}))_k \neq l_2$ and $(k(\mathbf{y}))_k \neq l_{n-1})$ or $((k(\mathbf{x}))_k \neq l'_2$ and $(k(\mathbf{y}))_k \neq l'_{n-1})$. By Lemma 4, there is a hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. Thus, we consider that $((k(\mathbf{x}))_k = l_2$ or $((k(\mathbf{y}))_k = l_{n-1})$ and $(k(\mathbf{x}))_k = l'_2$ or $(k(\mathbf{y}))_k = l'_{n-1})$. Since $l_2 \neq l'_2$ and $l_{n-1} \neq l'_{n-1}$, without loss of generality we assume that $(k(\mathbf{x}))_k = l_2$ and $(k(\mathbf{y}))_k = l'_{n-1}$.

Suppose that $|F(S_{n-1,1}^{l_1})| \geq 1$ or some edges in $\cup_{j\in\langle n\rangle-\{l_1\}} E^{l_1,j}$ are F-fault. Since $\cup_{j\in\langle n\rangle-\{l_1\}} E^{l_1,j} = n-1$ and $|F| \leq n-4$, there exists an index $i \in \langle n\rangle - \{l_1, l_2, l_n\}$ such that the only edge $(\mathbf{u}, k(\mathbf{u})) \in E^{l_1,i}$ is F-fault free. Since $(k(\mathbf{x}))_k = l_2 \neq i$, $\mathbf{u} \neq \mathbf{x}$. By Lemma 1, there exists a hamiltonian path P_6 of $S_{n-1,1}^{l_1} - F$ joining $\mathbf{x}$ to $\mathbf{u}$. Let $F' = F(S_{n-1,1}^{\langle n\rangle-\{l_1\}})$. Then $|R(F')| \leq n-5$. By Theorem 2, there exist two hamiltonian paths of $K_n^{\langle n\rangle-\{l_1\}} - R(F')$ joining i to l_n, say $P_3 = \langle i = a_1, a_2, \ldots, a_{n-1} = l_n\rangle$ and $P_4 = \langle i = b_1, b_2, \ldots, b_{n-1} = l_n\rangle$, such that $a_2 \neq b_2$ and $a_{n-2} \neq b_{n-2}$. Without loss of generality, we may assume that $(k(\mathbf{y}))_k \neq a_{n-2}$. Obviously, $(\mathbf{u})_k = l_1$. By Lemma 4, there exists a hamiltonian path P_5 of $S_{n-1,1}^{\langle n\rangle-\{l_1\}} - F'$ joining $k(\mathbf{u})$ to $\mathbf{y}$. Then $\langle \mathbf{x}, P_6, \mathbf{u}, k(\mathbf{u}), P_5, \mathbf{y}\rangle$ forms a hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Suppose that $|F(S_{n-1,1}^{l_1})| = 0$ and all edges in $\cup_{j\in\langle n\rangle-\{l_1\}} E^{l_1,j}$ are F-fault free. Let $\mathbf{u}$ be the only vertex of $S_{n-1,1}^{l_1}$ such that $(\mathbf{u}, k(\mathbf{u})) \in E^{l_1,l_n}$. Since $(k(\mathbf{y}))_k = l_{n-1} \neq (\mathbf{u})_k = l_1$ and $(k(\mathbf{x}))_k = l_2 \neq (k(\mathbf{u}))_k = l_n$, $k(\mathbf{u}) \neq \mathbf{y}$ and $\mathbf{u} \neq \mathbf{x}$. By Lemma 1, there exists a hamiltonian path P_7 of $S_{n-1,1}^{l_n} - F$ joining $k(\mathbf{u})$ to $\mathbf{y}$. Let $F' = F(S_{n-1,1}^{\langle n\rangle-\{l_1,l_n\}})$. Since $|F| \leq n-4$, $|R(F')| \leq n-4$. Thus, $K_n^{\langle n\rangle-\{l_1,l_n\}} - R(F')$ has a hamiltonian path, say $\langle c_1, c_2, \ldots, c_{n-2}\rangle$. Since all edges in $\cup_{j\in\langle n\rangle-\{l_1\}} E^{l_1,j}$ are F-fault free, the only edge $(\mathbf{v}, k(\mathbf{v}))$ in E^{l_1,c_1} and the only edge $(\mathbf{w}, k(\mathbf{w}))$ in $E^{l_1,c_{n-2}}$ are F-fault free. Obviously, $(k(k(\mathbf{v})))_k = (\mathbf{v})_k = l_1$ and $(k(k(\mathbf{w})))_k = (\mathbf{w})_k = l_1$. By Lemma 4, there exists a hamiltonian path P_8 of $S_{n-1,1}^{\langle n\rangle-\{l_1,l_n\}} - F'$ joining $k(\mathbf{v})$ to $k(\mathbf{w})$. Since $k(\mathbf{v})$ and $k(\mathbf{w})$ are the endpoints of the path P_8, at least one of $\mathbf{v}$ and $\mathbf{w}$ is not $\mathbf{x}$. Without loss of generality, we assume that $\mathbf{v} \neq \mathbf{x}$. Since $S_{n-1,1}^{l_1}$ is a complete graph, there exist two disjoint paths P_9 and P_{10} covering all vertices of $S_{n-1,1}^{l_1}$ such that P_9 joins $\mathbf{x}$ to $\mathbf{w}$ and P_{10} joins $\mathbf{u}$ to $\mathbf{v}$. Note that it is possible that $\mathbf{x}$ is $\mathbf{w}$. Hence, $\langle \mathbf{x}, P_9, \mathbf{w}, k(\mathbf{w}), P_8, k(\mathbf{v}), \mathbf{v}, P_{10}, \mathbf{u}, k(\mathbf{u}), P_7, \mathbf{y}\rangle$ forms a hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Case 2: $|F(S_{n-1,1}^1)| = n-4$. Thus, $S_{n-1,1}^1 - F$ is hamiltonian and $|F - F(S_{n-1,1}^1)| = 0$.

Subcase 2.1: $(\mathbf{x})_k = (\mathbf{y})_k = 1$. Choose any element f in $F(S_{n-1,1}^1)$. By Lemma 1, there exists a hamiltonian path P of $S_{n-1,1}^1 - F(S_{n-1,1}^1) + f$ joining $\mathbf{x}$ and $\mathbf{y}$. By deleting f, we can find two paths P_1 and P_2 covering all vertices of $S_{n-1,1}^1 - F$ such that P_1 joins $\mathbf{x}$ to $\mathbf{u}$, and P_2 joins $\mathbf{v}$ to $\mathbf{y}$. Since $S_{n-1,1}^1$ is a complete graph, $d(\mathbf{u},\mathbf{v}) = 1$. By Lemma 2, $(k(\mathbf{u}))_k \neq (k(\mathbf{v}))_k$. Obviously, $K_n^{\langle n\rangle - \{1\}}$ is hamiltonian connected and $(\mathbf{u})_k = (\mathbf{v})_k = 1$. With Lemma 4, there is a hamiltonian path P_3 of $S_{n-1,1}^{\langle n\rangle - \{1\}}$ joining $k(\mathbf{u})$ to $k(\mathbf{v})$. Thus, $\langle \mathbf{x}, P_1, \mathbf{u}, k(\mathbf{u}), P_3, k(\mathbf{v}), \mathbf{v}, P_2, \mathbf{y}\rangle$ forms a hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Subcase 2.2: $(\mathbf{x})_k = 1$ and $(\mathbf{y})_k \neq 1$. Let C be a hamiltonian cycle of $S_{n-1,1}^1 - F$. Write C as $\langle \mathbf{x}, \mathbf{u}, P_1, \mathbf{v}, \mathbf{x}\rangle$. By Lemma 2, $(k(\mathbf{u}))_k \neq (k(\mathbf{v}))_k$. Since the cycle C can be traversed forward and backward, we may assume that $(k(\mathbf{u}))_k = i \neq (\mathbf{y})_k$. Since $|F - F(S_{n-1,1}^1)| = 0$ and $n \geq 5$, there exists l in $\langle n\rangle - \{(\mathbf{x})_k, i, (\mathbf{y})_k\}$ such that the only edge $(\mathbf{w}, k(\mathbf{w}))$ in $E^{(\mathbf{y})_k, l}$ satisfies $\mathbf{w} \neq \mathbf{y}$. Obviously, $K_n^{\langle n\rangle - \{(\mathbf{x})_k, (\mathbf{y})_k\}}$ is hamiltonian connected, $(\mathbf{u})_k = (\mathbf{x})_k$, and $(\mathbf{w})_k = (\mathbf{y})_k$. With Lemma 4, there exists a hamiltonian path P_2 of $S_{n-1,1}^{\langle n\rangle - \{(\mathbf{x})_k, (\mathbf{y})_k\}}$ joining $k(\mathbf{u})$ to $k(\mathbf{w})$. By Lemma 1, there exists a hamiltonian path P_3 of $S_{n-1,1}^{(\mathbf{y})_k}$ joining $\mathbf{w}$ to $\mathbf{y}$. Thus, $\langle \mathbf{x}, \mathbf{v}, P_1, \mathbf{u}, k(\mathbf{u}), P_2, k(\mathbf{w}), \mathbf{w}, P_3, \mathbf{y}\rangle$ forms a hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Subcase 2.3: $(\mathbf{x})_k = (\mathbf{y})_k \neq 1$. By Lemma 1, there exists a hamiltonian cycle C of $S_{n-1,1}^1 - F$.

Assume that the only edge $(\mathbf{u}, k(\mathbf{u}))$ in $E^{(\mathbf{x})_k, 1}$ is F-fault free. Write C as $\langle k(\mathbf{u}), \mathbf{v}, P_2, \mathbf{v}', k(\mathbf{u})\rangle$. By Lemma 2, $(k(\mathbf{v}))_k \neq (k(\mathbf{v}'))_k$. Since $n \geq 5$, we can choose a vertex $\mathbf{w}$ in $S_{n-1,1}^{(\mathbf{x})_k}$ such that $(k(\mathbf{v}))_k \neq (k(\mathbf{w}))_k$, $\mathbf{w} \neq \mathbf{u}$, and $\mathbf{w} \neq \mathbf{y}$. Since $S_{n-1,1}^{(\mathbf{x})_k}$ is a complete graph, a hamiltonian path of $S_{n-1,1}^{(\mathbf{x})_k}$ can be written as $\langle \mathbf{x}, \mathbf{w}, P_4, \mathbf{y}\rangle$ if $\mathbf{x} = \mathbf{u}$ and $\langle \mathbf{x}, \mathbf{u}, \mathbf{w}, P_4, \mathbf{y}\rangle$ if $\mathbf{x} \neq \mathbf{u}$. Thus, such hamiltonian path can be expressed as $\langle \mathbf{x}, P_3, \mathbf{u}, \mathbf{w}, P_4, \mathbf{y}\rangle$. Obviously,

$K_n^{\langle n\rangle-\{1,(\mathbf{x})_k\}}$ is hamiltonian connected, $(\mathbf{v})_k = 1$, and $(\mathbf{w})_k = (\mathbf{x})_k$. By Lemma 4, there exists a hamiltonian path P_5 of $S_{n-1,1}^{\langle n\rangle-\{1,(\mathbf{x})_k\}}$ joining $k(\mathbf{v})$ to $k(\mathbf{w})$. Thus, $\langle \mathbf{x}, P_3, \mathbf{u}, k(\mathbf{u}), \mathbf{v}', P_2, \mathbf{v}, k(\mathbf{v}), P_5, k(\mathbf{w}), \mathbf{w}, P_4, \mathbf{y}\rangle$ forms a hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Assume that the only edge $(\mathbf{u}, k(\mathbf{u}))$ in $E^{(\mathbf{x})_k,1}$ is F-fault. Since $|\cup_{j\in\langle n\rangle-\{1\}} E^{1,j}| = n-1$ and $|F| \leq n-4$, there are at least three F-fault free edges in $\cup_{j\in\langle n\rangle-\{1\}} E^{1,j}$. Thus, there exists an index $r \in \langle n\rangle - \{(\mathbf{x})_k, 1\}$, such that the only edge $(\mathbf{v}, k(\mathbf{v}))$ in $E^{(\mathbf{x})_k,r}$ and the only edge $(\mathbf{w}, k(\mathbf{w}))$ in $E^{1,r}$ are F-fault free. Thus, $k(\mathbf{u}) \neq k(\mathbf{v})$. By Lemma 1, there exists a hamiltonian path P_1 of $S_{n-1,1}^r$ joining $k(\mathbf{v})$ to $k(\mathbf{w})$ and there exists a hamiltonian path P_2 of $S_{n-1,1}^{(\mathbf{x})_k}$ joining $\mathbf{x}$ to $\mathbf{y}$. We write P_2 as $\langle \mathbf{x}, \mathbf{v}, \mathbf{z}, P_4, \mathbf{y}\rangle$ and write C as $\langle \mathbf{w}, \mathbf{t}, P_2, \mathbf{t}', \mathbf{w}\rangle$. Since $S_{n-1,1}^1$ is a complete graph, $d(\mathbf{t}, \mathbf{t}') = 1$. By Lemma 2, $(k(\mathbf{t}))_k \neq (k(\mathbf{t}'))_k$. Since the cycle C can be traversed forward and backward, we may assume that $(k(\mathbf{t}))_k \neq (k(\mathbf{z}))_k$. Obviously, $K_n^{\langle n\rangle-\{(\mathbf{x})_k,1,r\}}$ is hamiltonian connected, $(\mathbf{t})_k = 1$, and $(\mathbf{z})_k = (\mathbf{x})_k$. By Lemma 4, there exists a hamiltonian path P_5 of $S^{\langle n\rangle-\{(\mathbf{x})_k,1,r\}}$ joining $k(\mathbf{t})$ to $k(\mathbf{z})$. Then $\langle \mathbf{x}, \mathbf{v}, k(\mathbf{v}), P_1, k(\mathbf{w}), \mathbf{w}, \mathbf{t}', P_2, \mathbf{t}, k(\mathbf{t}), P_5, k(\mathbf{z}), \mathbf{z}, P_4, \mathbf{y}\rangle$ forms a hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Subcase 2.4: $(\mathbf{x})_k, (\mathbf{y})_k$, and 1 are distinct. By Theorem 3, there exists a vertex set V' of $S_{n-1,1}^1$ with $|V'| \geq 3$ such that there exists a hamiltonian path of $S_{n-1,1}^1 - F$ joining every pair of vertices in V'. We define the F^* as $\{(1,l) \mid (\mathbf{u}, \mathbf{v}) \in E^{1,l} \text{ and } \mathbf{u} \notin V'\}$. Since $|V'| \geq 3$, $|F^*| \leq n-4$. By Theorem 2, there are two hamiltonian paths of $K_n - F^*$ joining $(\mathbf{x})_k$ to $(\mathbf{y})_k$, say $P_1 = \langle (\mathbf{x})_k = l_1, l_2, \ldots, l_n = (\mathbf{y})_k\rangle$ and $P_2 = \langle (\mathbf{x})_k = l_1', l_2', \ldots, l_n' = (\mathbf{y})_k\rangle$, such that $l_2 \neq l_2'$ and $l_{n-1} \neq l_{n-1}'$.

Suppose that $((k(\mathbf{x}))_k \neq l_2$ and $(k(\mathbf{y}))_k \neq l_{n-1})$ or $((k(\mathbf{x}))_k \neq l_2'$ and $(k(\mathbf{y}))_k \neq l_{n-1}')$. Without loss of generality, we assume that $(k(\mathbf{x}))_k \neq l_2$ and $(k(\mathbf{y}))_k \neq l_{n-1}$. Obviously, we can choose the only F-fault free edge $(\mathbf{v}^{l_t}, \mathbf{u}^{l_{t+1}})$ in $E^{l_t,l_{t+1}}$ for any $1 \leq t \leq n-1$. Since 1 is an internal vertex of P_1, we assume that $1 = l_i$. Since $k(\mathbf{u}^{l_i}) = \mathbf{v}^{l_{i-1}}$ and $k(\mathbf{v}^{l_i}) = \mathbf{u}^{l_{i+1}}$, $\mathbf{u}^{l_i} \neq \mathbf{v}^{l_i}$. Since $\mathbf{u}^{l_i}$ and $\mathbf{v}^{l_i}$ are in V' and $\mathbf{u}^{l_i} \neq \mathbf{v}^{l_i}$, there exists a hamiltonian path P_{l_i} of $S_{n-1,1}^{l_i} - F$ joining $\mathbf{u}^{l_i}$ to $\mathbf{v}^{l_i}$. Since $k(\mathbf{u}^{l_r}) = \mathbf{v}^{l_{r-1}}$ and $k(\mathbf{v}^{l_r}) = \mathbf{u}^{l_{r+1}}$, $\mathbf{u}^{l_r} \neq \mathbf{v}^{l_r}$. Since $S_{n-1,1}^{l_r}$ is a complete graph for any $l_r \in \langle n\rangle - \{1\}$ and $\mathbf{u}^{l_r} \neq \mathbf{v}^{l_r}$, there exists a hamiltonian path P_{l_r} of $S_{n-1,1}^{l_r}$ joining $\mathbf{u}^{l_r}$ to $\mathbf{v}^{l_r}$. Then $\langle \mathbf{x} = \mathbf{u}^{l_1}, P_{l_1}, \mathbf{v}^{l_1}, \mathbf{u}^{l_2}, P_{l_2}, \mathbf{v}^{l_2}, \ldots, \mathbf{u}^{l_n}, P_{l_n}, \mathbf{v}^{l_n} = \mathbf{y}\rangle$ forms a hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Thus, we consider that $((k(\mathbf{x}))_k = l_2$ or $(k(\mathbf{y}))_k = l_{n-1})$ and $((k(\mathbf{x}))_k = l'_2$ or $(k(\mathbf{y}))_k = l'_{n-1})$. Since $l_2 \neq l'_2$ and $l_{n-1} \neq l'_{n-1}$, without loss of generality we assume that $(k(\mathbf{x}))_k = l'_2$ and $(k(\mathbf{y}))_k = l_{n-1}$.

Suppose that there exists an index t such that $1 \leq t < n-2$, $1 \neq l_t$, and $1 \neq l_{t+1}$. Since $|F - F(S^1_{n-1,1})| = 0$, the only edge $(\mathbf{p}, k(\mathbf{p}))$ in E^{l_n,l_t} and the only edge $(\mathbf{q}, k(\mathbf{q}))$ in $E^{l_n,l_{t+1}}$ are F-fault free. Obviously, $P_3 = \langle l_1, \cdots, l_t, l_n, l_{t+1}, \ldots, l_{n-1}\rangle$ is also a hamiltonian path of $K_n - R(F^*)$. We rewrite P_3 as $\langle a_1, a_2, \ldots, a_n\rangle$. We set $\mathbf{x} = \mathbf{u}^{a_1}$ and $k(\mathbf{y}) = \mathbf{v}^{a_n}$. Then we choose the only edge in $E^{a_r,a_{r+1}}$ as $(\mathbf{v}^{a_r}, \mathbf{u}^{a_{r+1}})$ for any $1 \leq r \leq n$. Since $k(\mathbf{v}^{a_r}) = \mathbf{u}^{a_{r+1}}$ and $k(\mathbf{u}^{a_r}) = \mathbf{v}^{a_{r-1}}$, $\mathbf{v}^{a_r} \neq \mathbf{u}^{a_r}$. Let i be the index such that $a_i = 1$. Obviously, $\mathbf{u}^{a_i}$ and $\mathbf{v}^{a_i}$ are in V' and there exists a hamiltonian path P_{a_i} of $S^1_{n-1,1} - F$ joining $\mathbf{u}^{a_i}$ to $\mathbf{v}^{a_i}$. Since $S^r_{n-1,1}$ is complete graph for any $r \in \langle n\rangle$, there exists a hamiltonian path P_{a_r} of $S^{a_r}_{n-1,1} - \{\mathbf{y}\}$ joining $\mathbf{u}^{a_r}$ to $\mathbf{v}^{a_r}$ for any $a_r \in \langle n\rangle - \{1\}$. Then $P_4 = \langle \mathbf{x} = \mathbf{u}^{a_1}, P_{a_1}, \mathbf{v}^{a_1}, \mathbf{u}^{a_2}, P_{a_2}, \mathbf{v}^{a_2}, \ldots, \mathbf{u}^{a_n}, P_{a_n}, \mathbf{v}^{a_n} = k(\mathbf{y})\rangle$ forms a hamiltonian path of $S_{n,2} - F - \{\mathbf{y}\}$ joining $\mathbf{x}$ to $k(\mathbf{y})$. Then $\langle \mathbf{x}, P_4, k(\mathbf{y}), \mathbf{y}\rangle$ forms a hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Suppose that there is no index t such that $1 \leq t < n-2$, $1 \neq l_t$, and $1 \neq l_{t+1}$. We claim that $n = 5$ and $l_2 = 1$. Let p be the index such that $l_p = 1$. Suppose that $n \geq 6$. We can choose t to be 1 if $p \geq 3$ and 3 if otherwise. Suppose that $n = 5$ and $l_2 \neq 1$. We can choose t to be 1. Obviously $1 \neq l_t$ and $1 \neq l_{t+1}$. We get a contradiction.

Thus, we only consider the case that $n = 5$ and $l_2 = 1$. Since $(k(\mathbf{x}))_k = l'_2 \neq l'_5 = l_5$ and $(k(\mathbf{y}))_k = l_4 \neq l_1$, the only edge $(\mathbf{u}, k(\mathbf{u})) \in E^{l_1,l_5}$ satisfies $\mathbf{u} \neq \mathbf{x}$ and $k(\mathbf{u}) \neq \mathbf{y}$. By Lemma 1, there exists a hamiltonian path P_7 of $S^{l_5}_{n-1,1}$ joining $k(\mathbf{u})$ to $\mathbf{y}$. Since $|F - F(S^1_{n-1,1})| = 0$, the only edge $(\mathbf{w}, k(\mathbf{w}))$ in E^{l_1,l_4} is F-fault free. Since $(l_1, l_2) \in P_1$, the only edge $(\mathbf{v}, \mathbf{v}^{l_2})$ in E^{l_1,l_2} is F-fault free. Since $S^{l_1}_{n-1,1}$ is a complete graph, there exist two paths P_5 and P_6 covering all vertices in $S^{l_1}_{n-1,1}$ such that P_5 joins $\mathbf{x}$ to $\mathbf{w}$ and P_6 joins $\mathbf{v}$ to $\mathbf{u}$. Since $\langle l_2, l_3, l_4\rangle$ is a subpath of P_1, the only edge $(\mathbf{u}^{l_2}, \mathbf{v}^{l_3})$ in E^{l_2,l_3} and the only edge $(\mathbf{u}^{l_3}, \mathbf{v}^{l_4})$ in E^{l_3,l_4} are F-fault free. Since $\mathbf{v}^{l_2}$ and $\mathbf{u}^{l_2}$ are in V', by Lemma 3 there exists a hamiltonian path P_2 of $S^{l_2}_{n-1,1} - F$ joining $\mathbf{u}^{l_2}$ to $\mathbf{v}^{l_2}$. By Lemma 1, there exists a hamiltonian path P_{l_3} of $S^{l_3}_{n-1,1}$ joining $\mathbf{u}^{l_3}$ to $\mathbf{v}^{l_3}$ and a hamiltonian path P_{l_4} of $S^{l_2}_{n-1,1}$ joining $k(\mathbf{w})$ to $\mathbf{v}^{l_4}$. Then $\langle \mathbf{x}, P_5, \mathbf{w}, k(\mathbf{w}), P_{l_4}, \mathbf{v}^{l_4}, \mathbf{u}^{l_3}, P_{l_3}, \mathbf{v}^{l_3}, \mathbf{u}^{l_2}, P_{l_2}, \mathbf{v}^{l_2}, \mathbf{v}, P_6, \mathbf{u}, k(\mathbf{u}), P_7, \mathbf{y}\rangle$ forms a hamiltonian path of $S_{5,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Thus, the lemma is proved. $\square$

Lemma 8 *Suppose that, for some $k \geq 3$ and $n-k \geq 2$, $S_{n-1,k-1}$ is $(n-4)$-fault hamiltonian and $(n-5)$-fault hamiltonian connected. Then $S_{n,k}$ is $(n-4)$-fault hamiltonian connected.*

Proof. Since $k \geq 3$, $n \geq 5$, and $(n-k) \geq 2$, $|E^{r,s}| = \frac{(n-2)!}{(n-k)!} \geq (n-2)$ for all $1 \leq r \neq s \leq n$. By Lemma 3, all edges in $E^{r,s}$ are independent. Assume that F is any faulty set of $S_{n,k}$ with $|F| \leq n-4$. Without loss of generality, we assume that $|F(S^1_{n-1,k-1})| \geq |F(S^2_{n-1,k-1})| \geq \cdots \geq |F(S^n_{n-1,k-1})|$. Let $\mathbf{x}$ and $\mathbf{y}$ be any two arbitrary vertices of $S_{n,k} - F$. We want to construct a hamiltonian path of $S_{n,k} - F$ joining $\mathbf{x}$ and $\mathbf{y}$.

Case 1: $|F(S^1_{n-1,k-1})| \leq n-5$. By the assumption of this lemma, $S^t_{n-1,k-1} - F(S^t_{n-1,k-1})$ is hamiltonian connected for any $t \in \langle n \rangle$.

Subcase 1.1: $(\mathbf{x})_k \neq (\mathbf{y})_k$. Since $|R(F)| \leq n-4$, by Lemma 1 $K_n - R(F)$ is hamiltonian connected. By Lemma 4, there exists a hamiltonian path of $S_{n,k} - F$ joining $\mathbf{x}$ and $\mathbf{y}$.

Subcase 1.2: $(\mathbf{x})_k = (\mathbf{y})_k$. By the assumption of this lemma, there exists a hamiltonian path P_1 of $S^{(\mathbf{x})_k}_{n-1,k-1} - F(S^{(\mathbf{x})_k}_{n-1,k-1})$ joining $\mathbf{x}$ to $\mathbf{y}$. We claim that there exists an edge $(\mathbf{u}, \mathbf{v})$ of P_1 such that $(\mathbf{u}, k(\mathbf{u}))$ in $E^{(\mathbf{x})_k,(k(\mathbf{u}))_k}$ and $(\mathbf{v}, k(\mathbf{v}))$ in $E^{(\mathbf{x})_k,(k(\mathbf{v}))_k}$ are F-fault free. Let F' denote the the set of F-fault edge in $\cup_{j \in \langle n \rangle - \{(\mathbf{x})_k\}} E^{(\mathbf{x})_k,j}$. Suppose that no such edge exists. Then $|F'| \geq |P_1|/2$. Thus, $|F' \cup F(S^{(\mathbf{x})_k}_{n-1,k-1})| \geq \frac{(n-1)!}{2(n-k)!} > |F|$ when $n \geq 5$ and $k \geq 3$. We get a contradiction.

Thus, we can write P_1 as $\langle \mathbf{x}, P_2, \mathbf{u}, \mathbf{v}, P_3, \mathbf{y} \rangle$. Since $d(\mathbf{u}, \mathbf{v}) = 1$, $(k(\mathbf{u}))_k \neq (k(\mathbf{v}))_k$. Let $\langle (k(\mathbf{u}))_k = l_1, l_2, \ldots, l_{n-1} = (k(\mathbf{v}))_k \rangle$ be any hamiltonian path of $K_n^{\langle n \rangle - \{(\mathbf{x})_k\}}$. We set $k(\mathbf{u}) = \mathbf{v}^{l_1}$ and $k(\mathbf{v}) = \mathbf{u}^{l_{n-1}}$. Since $|E^{r,s}| - |F| \geq 2$ for any $r, s \in \langle n \rangle$, we can choose any F-fault free edges $(\mathbf{u}^{l_i}, \mathbf{v}^{l_{i+1}})$ in $E^{l_i,l_{i+1}}$ for all $1 \leq i \leq n-1$. By the assumption of this lemma, there exists a hamiltonian path P_{l_i} of $S^{l_i}_{n-1,k-1} - F$ joining $\mathbf{v}^{l_i}$ to $\mathbf{u}^{l_i}$. Then, $P_4 = \langle \mathbf{v}^{l_1}, P_{l_1}, \mathbf{u}^{l_1}, \mathbf{v}^{l_2}, \ldots, P_{l_{n-1}}, \mathbf{u}^{l_{n-1}} \rangle$ forms a hamiltonian path of $S^{\langle n \rangle - \{(\mathbf{x})_k\}}_{n-1,k-1} - F$ joining $k(\mathbf{u})$ to $k(\mathbf{v})$. Thus, $\langle \mathbf{x}, P_2, \mathbf{u}, k(\mathbf{u}), P_4, k(\mathbf{v}), \mathbf{v}, P_3, \mathbf{y} \rangle$ forms a hamiltonian path of $S_{n,k} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Case 2: $|F(S^1_{n-1,k-1})| = n-4$.

Subcase 2.1: $(\mathbf{x})_k = (\mathbf{y})_k = 1$. Choose any element f in $F(S^1_{n-1,k-1})$. By the assumption of this lemma, we can find a hamiltonian path P of $S^1_{n-1,k-1} - F(S^1_{n-1,k-1}) + f$ joining $\mathbf{x}$ to $\mathbf{y}$. By deleting f, we can find two vertices $\mathbf{u}$ and $\mathbf{v}$ with $d(\mathbf{u},\mathbf{v}) \leq 2$ such that (1) there are two paths P_1 and P_2 covering all the vertices of $S^1_{n-1,k-1} - F$, (2) P_1 joins $\mathbf{x}$ to $\mathbf{u}$, and (3) P_2 joins $\mathbf{v}$ to $\mathbf{y}$. By Lemma 2, $(k(\mathbf{u}))_k \neq (k(\mathbf{v}))_k$. Then there exists a hamiltonian path of $K_n^{\langle n\rangle - \{1\}}$ joining $(k(\mathbf{u}))_k$ to $(k(\mathbf{v}))_k$. By Lemma 4, there is a hamiltonian path P_3 joining $k(\mathbf{u})$ and $k(\mathbf{v})$ in $S^{\langle n\rangle-\{1\}}_{n-1,k-1}$. Thus, $\langle \mathbf{x}, P_1, \mathbf{u}, k(\mathbf{u}), P_3, k(\mathbf{v}), \mathbf{v}, P_2, \mathbf{y}\rangle$ forms a hamiltonian path of $S_{n,k} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Subcase 2.2: $(\mathbf{x})_k = 1$ and $(\mathbf{y})_k \neq 1$. Let C be a hamiltonian cycle of $S^1_{n-1,k-1} - F(S^1_{n-1,k-1})$. Write C as $\langle \mathbf{x}, \mathbf{u}, P_1, \mathbf{v}, \mathbf{x}\rangle$. Thus, $d(\mathbf{u},\mathbf{v}) \leq 2$. By Lemma 2, $(k(\mathbf{u}))_k \neq (k(\mathbf{v}))_k$. Since the cycle C can be traversed forward and backward, we may assume that $(k(\mathbf{u}))_k \neq (\mathbf{y})_k$. Since $|F - F(S^1_{n-1,k-1})| = 0$ and $|E^{r,s}| \geq (n-2)$ for any $1 \leq r < s \leq n$, there exists an F-fault free edge $(\mathbf{w}, k(\mathbf{w}))$ in $E^{(\mathbf{y})_k,l}$ for some $l \in \langle n\rangle - \{(\mathbf{x})_k, (k(\mathbf{u}))_k, (\mathbf{y})_k\}$ such that $\mathbf{w} \neq \mathbf{y}$. Obviously, there exists a hamiltonian path of $K_n^{\langle n\rangle - \{(\mathbf{x})_k,(\mathbf{y})_k\}}$ joining $(k(\mathbf{u}))_k$ to $(k(\mathbf{w}))_k$, $(k(k(\mathbf{u})))_k = (\mathbf{u})_k = (\mathbf{x})_k$, and $(\mathbf{w})_k = (\mathbf{y})_k$. By Lemma 4, there exists a hamiltonian path P_2 of $S^{\langle n\rangle - \{(\mathbf{x})_k,(\mathbf{y})_k\}}_{n-1,k-1}$ joining $k(\mathbf{u})$ to $k(\mathbf{w})$. With the assumption of this lemma, there exists a hamiltonian path P_3 of $S^{(\mathbf{y})_k}_{n-1,k-1}$ joining $\mathbf{w}$ to $\mathbf{y}$. Then $\langle \mathbf{x}, \mathbf{v}, P_1, \mathbf{u}, k(\mathbf{u}), P_2, k(\mathbf{w}), \mathbf{w}, P_3, \mathbf{y}\rangle$ forms a hamiltonian path of $S_{n,k} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Subcase 2.3: $(\mathbf{x})_k = (\mathbf{y})_k \neq 1$. With the assumption of this lemma, there exists a hamiltonian path P_1 of $S^{(\mathbf{x})_k}_{n-1,k-1}$ joining $\mathbf{x}$ to $\mathbf{y}$ and there exists a hamiltonian cycle C of $S^1_{n-1,k-1} - F(S^1_{n-1,k-1})$. Since $|E^{(\mathbf{x})_k,1}| - |F| \geq 2$, there exists an F-fault free edge $(\mathbf{u}, k(\mathbf{u})) \in E^{(\mathbf{x})_k,1}$ with $\mathbf{x} \neq \mathbf{u}$. Thus, we can write C as $\langle k(\mathbf{u}), \mathbf{v}, P_2, \mathbf{v}', k(\mathbf{u})\rangle$ and write P_1 as $\langle \mathbf{x}, P_3, \mathbf{u}, \mathbf{w}, P_4, \mathbf{y}\rangle$. Since $d(\mathbf{v},\mathbf{v}') \leq 2$, $(k(\mathbf{v}))_k \neq (k(\mathbf{v}'))_k$. Without loss of generality, we assume that $(k(\mathbf{v}))_k \neq (k(\mathbf{w}))_k$. Obviously, there exists a hamiltonian path of $K_n^{\langle n\rangle - \{1,(\mathbf{x})_k\}}$ joining $(k(\mathbf{u}))_k$ to $(k(\mathbf{w}))_k$. By Lemma 4, there exists a hamiltonian path P_5 of $S^{\langle n\rangle - \{1,(\mathbf{x})_k\}}_{n-1,k-1}$ joining $k(\mathbf{v})$ to $k(\mathbf{w})$. Thus, $\langle \mathbf{x}, P_3, \mathbf{u}, k(\mathbf{u}), \mathbf{v}', P_2, \mathbf{v}, k(\mathbf{v}), P_5, k(\mathbf{w}), \mathbf{w}, P_4, \mathbf{y}\rangle$ forms a hamiltonian path of $S_{n,k} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Subcase 2.4: $(\mathbf{x})_k, (\mathbf{y})_k$, and 1 are distinct. Since $|E^{(\mathbf{x})_k,1}| \geq (n-2)$, there exists an F-fault free edge $(\mathbf{u}, k(\mathbf{u}))$ in $E^{(\mathbf{x})_k,1}$. Let C be a hamiltonian cycle of $S^1_{n-1,k-1} - F$. We can write C as $\langle k(\mathbf{u}), \mathbf{v}, P_1, \mathbf{w}, k(\mathbf{u})\rangle$. Since $d(\mathbf{v},\mathbf{w}) \leq 2$, $(k(\mathbf{v}))_k \neq (k(\mathbf{w}))_k$. Without loss of generality, we assume that $(\mathbf{v})_k \neq (\mathbf{y})_k$. Then there exists a hamiltonian path of $K_n^{\langle n\rangle-\{(\mathbf{x})_k,1\}}$ joining $(\mathbf{v})_k$ to $(\mathbf{y})_k$. By Lemma 4, there exists a hamiltonian path P_2 of $S^{\langle n\rangle-\{(\mathbf{x})_k,1\}}$ joining $\mathbf{v}$ to $\mathbf{y}$. By the assumption of this lemma, there exists a hamiltonian path P_3 of $S_{n-1,k-1}^{(\mathbf{x})_k}$ joining $\mathbf{x}$ to $\mathbf{u}$. Then $\langle \mathbf{x}, P_3, \mathbf{u}, k(\mathbf{u}), k(\mathbf{w}), P_1, k(\mathbf{v}), \mathbf{v}, P_2, \mathbf{y}\rangle$ forms a hamiltonian path of $S_{n,k} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Thus, the lemma is proved. □

Theorem 4 *Let n and k be two positive integers with $n > k \geq 1$. Then*

(1) $\mathcal{H}_f(S_{n,k}) = n-3$ and $\mathcal{H}_f^\kappa(S_{n,k}) = n-4$ if $n-k \geq 2$;

(2) $\mathcal{H}_f(S_{2,1})$ is undefined and $\mathcal{H}_f^\kappa(S_{2,1}) = 0$; and

(3) $\mathcal{H}_f(S_{n,n-1}) = 0$ and $\mathcal{H}_f^\kappa(S_{n,n-1})$ is undefined if $n > 2$.

Proof. We first consider the case $k = n-1$. It is proved in [3] that $S_{n,n-1}$ is isomorphic to the star graph S_n. The star graph is a bipartite graph. It is proved in [1] that S_n is hamiltonian if and only if $n > 2$. The star graph S_2 is K_2 which is hamiltonian connected. It is known that the number of vertices in both partite sets of any bipartite hamiltonian are the same. For these reasons, any bipartite hamiltonian graph is not hamiltonian connected. Thus, $\mathcal{H}_f(S_{n,n-1}) = 0$ and $\mathcal{H}_f^\kappa(S_{n,n-1})$ is undefined if $n > 2$. Moreover, $\mathcal{H}_f(S_{2,1})$ is undefined and $\mathcal{H}_f^\kappa(S_{2,1}) = 0$.

Now, we consider the case $n > k \geq 1$. By Lemma 1, the theorem is true for $S_{n,1}$. According to Lemma 5, the theorem is true for $S_{4,2}$. Based on the Lemmas 6 and 7, the theorem is true for $S_{n,2}$ with $n \geq 5$. By Lemmas 6 and 8, the theorem is true for all $S_{n,k}$ with $n \geq 5$, $k \geq 3$, and $(n-k) \geq 2$. Hence, the theorem is proved. □

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