A Study of the Generalized Petersen Graph and a Novel Graph Y

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Abstract—This paper presents a detailed study of several properties of the generalized Petersen graph, which are considered important to the parallel architectures. Similar to the well-known Petersen graph, the generalized Petersen graph contains two cycles; all the nodes in one cycle are each linked to their counterparts in the other cycle. Each cycle can hold either exactly five nodes or any number of nodes greater than six, according to our study. Besides the detailed analyses of several important properties of the generalized Petersen graph, we also propose a shortest-path routing algorithm and a general method for its VLSI layout.

The second major contribution of this paper is the proposition of a new graph, denoted as Y, developed in conjunction with a novel approach to graph expansion. This expansion method recursively utilizes the structure of a bipartite graph and substitutes each node with a cyclic supernode. When compared with the popular hypercube structure, the Y graph not only contains up to 25% more nodes, but also has smaller diameters. Furthermore, close comparisons with the generalized Petersen graph reveals that they are both regular, but the Y graph is more favorable for being less restrictive in properties like number of nodes, degree, and diameter. Its structural flexibility can easily be seen from the fact that the generalized Petersen graph can be derived from the Y graph with ease. We also develop for the Y graph a shortest-path routing algorithm.

Index Terms—Generalized Petersen graph, Y graph, interconnection network, routing algorithm, VLSI layout, graph theory, parallel architecture, distributed system, self-organization.

1 INTRODUCTION

The study of interconnection networks has always been an important area of research for multi processor and/or parallel computer systems. The interconnection networks can be viewed as graphs composed of node sets and edge sets. The node can be designated as processors, while the edge can be viewed as the communication links to each node.

The conditions to evaluate the interconnection networks include diameter, degree, regularity, symmetry, and so on. Desirable properties of an interconnection network include low degree, low diameter, symmetry, low congestion, high connectivity, high fault tolerance, and efficient routing algorithms. For example, graphs with small diameters and/or small vertex degrees are well suited for massively parallel computation. A small vertex degree implies that the system can be implemented with lower hardware cost for communication. A fixed vertex degree implies that the system can be expanded without having to modify the structure of the individual nodes. However, in most exiting interconnection networks, these requirements are often in conflict with each other.

There is a close relationship between the efficiency of a parallel computer system and the efficiency of the interconnection network [13] it relies on. Thus, there have been numerous interconnection networks proposed, including hypercubes, twisted hypercubes, pyramids, cube-connected cycles (CCC) [12] and so on.

A network based on regular graph [14] is particularly suited to parallel computers. It follows that the Peterson graph [2] has been extensively studied and many of its extensions have been proposed, such as *folded Petersen networks* [10], *folded Petersen cube networks*, and *hyper Petersen networks* [4].

This paper first analyzes some properties of the generalized Petersen graphs (GP) pertinent to parallel architectures. Furthermore, a new class of topology based on the bipartite graph, called the Y graphs, is proposed and analyzed in this paper.

In Section 2, we discuss the properties of GP in detail. The shortest-path routing algorithms for GP are also included. Section 3 defines the Novel Y graphs. We also analyzed their topological and performance properties. In Section 4, we discuss the embedding and VLSI layout on the proposed networks. Section 5 concludes the paper.

2 GENERALIZED PETERSEN GRAPHS

The Petersen graph can be viewed as two cycles, and between them, all the nodes in one of the cycles will respectively conform to the regulations and link to their corresponding nodes in another cycle. GP, in the same way, come in two cycles in which the numbers of each cycle's nodes are identical, given they are 5 or any given integer greater than 6.

2.1 THE PETERSEN GRAPH AND ITS EXPANSIONS

The well-known Petersen graph (see Figure 1) is the simplest case of GP, GP(5, 2), which has 10 nodes of degree 3 with outer 5-cycle, inner 5-cycle, and five links joining them. The Petersen graph is symmetric, a diameter 3, and the most efficient small network in terms of node degree, diameter, and network size. Due to the Petersen graph's unique and optimal properties, several network topologies based on the Petersen graph have been proposed and investigated in research literature [4][8][10]. We give the Petersen Graph a mathematical definition as follows.

DEFINITION 1. GP(5, 2) $= \{V, E\}$, $V = \{v_{i,j} \mid i = 0 \text{ or } 1, j = 0, 1, 2, 4\}$, $E = \{ v_{0,j}\, v_{0,(j+1) \bmod 5}$, $v_{1,j}\, v_{1,(j+1) \bmod 5}$, $v_{1,j}\, v_{0,(5w+j)/2}$, where w is the smallest nonnegative integer such that $(5w + j)$ is divisible by 2, $j = 0, 1, 2, 4\}$.

Figures 2 and 3 are two expansions of the Petersen graph [3]. The folded Petersen graph in Figure 2 can also be viewed as the graphic production expansion, shown as Figure 4. The root-folded Petersen graph in Figure 3 is decreasing the edges of the folded Petersen graph. It makes each supernode have only 1 node linking to other supernodes with the purpose of decreasing cost, yet flawed because it causes asymmetric graphs.

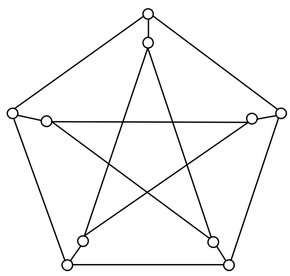
Figure 1. Petersen graph.

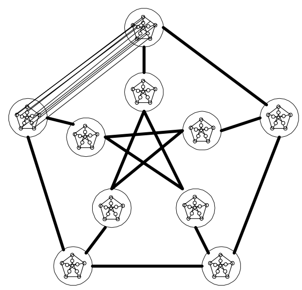
Figure 2. Folded Petersen graph.

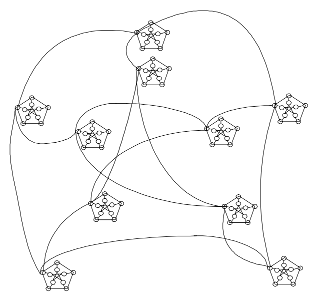
Figure 3. Root-folded Petersen graph.

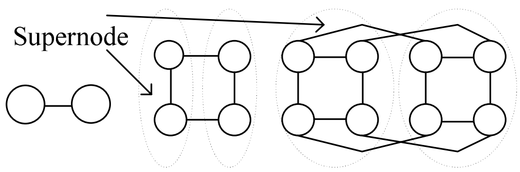

Figure 4. The production of graphs.

2.2 THE DEFINITION FOR GENERALIZED PETERSEN GRAPHS

The order of nodes in cycle C_8 (see Figure 5) will be addressed as 0, 1, 2, 3, 4, 5, 6, and 7. If we start with the node 0, according to the order, linking every other node can enclose a cycle C_4 (see Figure 6). If it is to link with interval of three nodes, it will create another cycle C_8 (see Figure 7).

When we link nodes with the interval of k nodes in C_m where m is divisible by k, it will not create another C_m; if m is indivisible by k, it will create a new C_m. Since linking with interval of k nodes and $m - k$ nodes will create the same result, we can qualify the k of GP(m, k) as min (k, $m - k$) or regulated $k < m/2$.

We detail the definition of the generalized Petersen graphs as definition 2.

DEFINITION 2. GP(m, k) $= \{V, E\}$, $V = \{v_{i,j} \mid i = 0 \text{ or } 1, j = 0, 1, \ldots, m-1\}$, $E = \{ v_{0,j}\,v_{0,(j+1) \bmod m}, v_{1,j}\,v_{1,(j+1) \bmod m}, v_{1,j}\,v_{0,(m\times w+j)/k}$, where w is the smallest nonnegative integer such that $(m\times w + j)$ is divisible by k, $j = 0, 1, \ldots, m-1\}$, $1 < k < m/2$, m and k are relatively prime integers.

GP(8, 3) (see Figure 8) and Petersen graphs GP(5, 2) are the special cases of GP. In GP, $v_{1,j}$ and $v_{0,j}$ are designated as the nodes in the outer and inner cycles respectively. According to definition 2, the size of node set V is $2m$. In the edge set E of definition 2, $v_{0,j}\,v_{0,(j+1) \bmod m}$ and $v_{1,j}\,v_{1,(j+1) \bmod m}$ are designated as the edge of the inner and outer cycles respectively. The relation of the two cycles in GP(8, 3) is shown in Figure 9.

According to definition 2, the second coordinate of the address in the inner node linking to node $v_{1,j}$ is $(m\times w + j)/k$. In Figure 9, $m = 8$, $k = 3$ and in the 1st row of the inner cycle w and $(m\times w + 3)/k$ equal to 0 and 1, respectively, thus $v_{1,3}$ and $v_{0,1}$ are next to each other.

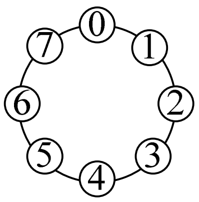

Figure 5. Cycle C_8.

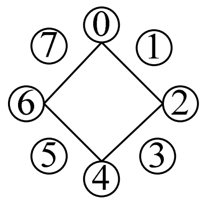

Figure 6. Cycle C_8 turns into cycle C_4.

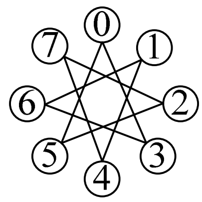

Figure 7. Another cycle C_8.

2.3 THE EXISTENCE OF GP

In Figure 10, it is indicated the interconnection of GP similar to Figure 9. According to the definition of GP, the graphs with different number of nodes are derived (see Table 1).

Table 1. The Relation between the Number of Graphs in GP(m, k) and m

The m of GP(m, k)	5	7	8	9	Prime integer, m
The k of GP(m, k)	2	2, 3	3	2, 4	2~$\lfloor (m-1)/2 \rfloor$
Number of Graphs	1	2	1	2	$\lfloor (m-3)/2 \rfloor$

The D($v_{i,j}$, $v_{i',j'}$) is to designate the shortest distance between $v_{i,j}$ and $v_{i',j'}$, and the Dia(G) is defined as the diameter of graph G.

LEMMA 1. For a given m, the numbers of graphs in GP(m, k) is equal to or less then $\lfloor (m-3)/2 \rfloor$. When the numbers of graphs in GP(m, k) is equal to $\lfloor (m-3)/2 \rfloor$, m is a prime.

Proof: According to definition 2, the conditions will be $1 < k < m/2$; m and k are relatively primes. When m is a prime, $k = 2, 3\ldots, \lfloor (m-1)/2 \rfloor$ and m are relatively prime. Thus, there are as many as $\lfloor (m-3)/2 \rfloor$ of the numbers of graphs in GP(m, k). When m is a prime, m is indivisible by any k. For a given m, then the numbers of graphs in GP(m, k) will not be greater than that when m is a prime. This concludes the proof.■

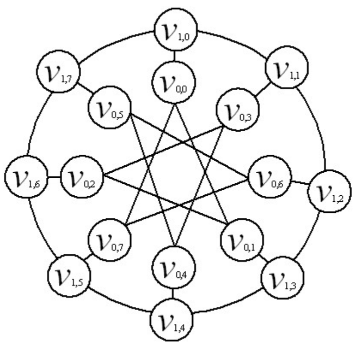

Figure 8. GP(8, 3).

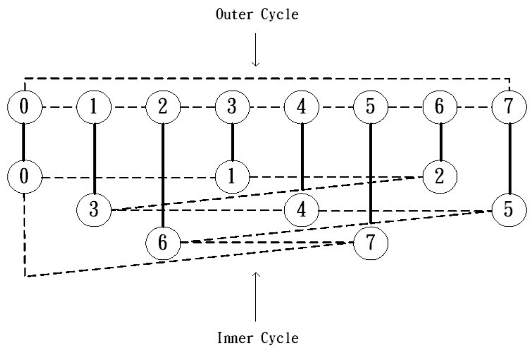

Figure 9. The relation of two cycles of GP(8, 3)

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Outer Cycle	0	1	2	3	4	5	6	7
	0			1			2	
Inner Cycle		3			4			5
			6			7		

Figure 10. Another expression of GP(8, 3)

Table 2. The Two Greatest Numbers in the Range of k Values

Even number, m	10	12	14	m
Two largest numbers, range k	3, 4	4, 5	5, 6	$(m-4)/2$, $(m-2)/2$

THEOREM 1. If m is an integer which is equal to 5 or greater than 6, then GP(m, k) will exist.

Proof: When m is an odd number, we can assign $k = 2$ and make GP(m, k) exist. Now we only need to show that GP(m, k) exists when m is an even number. When m is an even number, we assume that $m = 2x$, where x is an integer. According to definition 2 the two greatest numbers in the range of k values are $(m-4)/2$ and $(m-2)/2$, namely $x-2$ and $x-1$, as shown in Table 2.

When x is an even number, $x-1$ is an odd number. We could let $k = x-1$, since odd numbers and 2 are relatively prime and any two consecutive integers are relatively prime that we can see by Euclid's algorithm. Since k and m are relatively primes, GP(m, k) exists.

When x is an odd number, $x - 2$ is also an odd number. We could let $k = x - 2$ for the same reason mentioned above. Therefore GP(m, k) exists since k and m are relatively prime. According to definition 2, $k > 1$, when $m = 5$ or $m > 6$, we can find some k values by which m is indivisible. Therefore GP(m, k) exists where m is equal to 5 or greater than 6.■

2.4 SYMMETRIC PROPERTY

The simplified expression of GP(7, 2) will be shown as Figure 11. The result of the two cycles exchange in GP(7, 2) is shown in Figure 12. Since the total number of nodes in each cycle is 7, when we view it from another direction, it will link to form the graph GP(7, 7 – 4). We deduce the two following points: (1) GP(7, 2) and GP(7, 3) are isomorphic and (2) GP(7, 2) is not a symmetric graph. When we are viewing from the outer cycle toward the inner cycle, the inner cycle is linked with the interval of 2 nodes. While under the reverse situation, its inner cycle is linked with the interval of 3 nodes. However, GP(8, 3) is featured with symmetric properties, as shown in Figures 13 and 14.

Outer Cycle	0	1	2	3	4	5	6
Inner Cycle	0	4	1	5	2	6	3

Figure 11. GP(7, 2)

Outer Cycle	0	1	2	3	4	5	6
Inner Cycle	0	2	4	6	1	3	5

Figure 12. Two cycle exchange in GP(7, 2)

Outer Cycle	0	1	2	3	4	5	6	7
Inner Cycle	0	3	6	1	4	7	2	5

Figure 13. GP(8, 3)

Outer Cycle	0	3	6	1	4	7	2	5
Inner Cycle	0	1	2	3	4	5	6	7

Figure 14. Two cycle exchange in GP(8, 3)

LEMMA 2. GP(m, k) and GP(m, min(k_{iso}, $m - k_{iso}$)) respectively are isomorphic graphs of each other. When k = min(k_{iso}, $m - k_{iso}$), GP(m, k) is a symmetric graph. Among them, $k_{iso} = (m \times w + 1)/k$, w is the smallest nonnegative integer where $(m \times w + 1)$ is divisible by k.

Proof: According to the definition of GP(m, k), $v_{1,\,k}$ and $v_{0,\,1}$ are adjacent nodes. If $v_{0,\,k_{iso}}$ and $v_{1,\,1}$ are adjacent nodes, then after the exchange of inner and outer cycles, $v_{1,}$

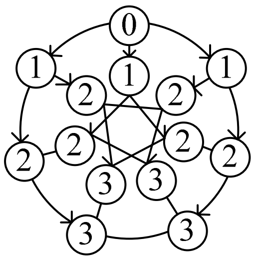
k_{iso} and $v_{0,1}$ will also be adjacent nodes. In addition, according to definition 2 and the description of this property, in GP(m, k), $v_{0,k_{iso}}$ and $v_{1,1}$ are adjacent nodes, which form GP(m, min(k_{iso}, $m - k_{iso}$)). Furthermore, since k = min(k_{iso}, $m - k_{iso}$) means the identical k value after the exchange of two cycles, and the individual nodes in a cycle have already come in the symmetric property, when k = min(k_{iso}, $m - k_{iso}$), GP(m, k) is a symmetric graph.■

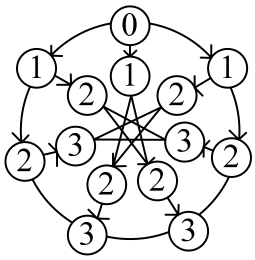
2.5 Distances of GP

According to the graphs discussed previously, we can deduce that the numbers of graphs in GP(m, k) will be infinite as the m's increase. We will analyze the diameters and distances of these graphs in this section.

Graphs like Figure 8 can be adopted to indicate pulsing with the designation of distance, as shown in Figures 15, 16 and 17. Among them, $v_{1,0}$ is selected as the source node with the distance of 0; therefore the distance of its adjacent nodes is 1.

In Table 3, $v_{0,1}$ is adjacent to $v_{0,0}$ with its distance of 1. We find that the diameter of GP(8, 3) is 4 and the average distance of any 2 given nodes is 2.125.





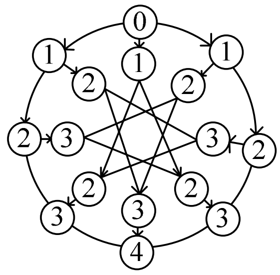


Figure 15. The Distance of GP(7, 2). Figure 16. The Distance of GP(7, 3). Figure 17. The Distance of GP(8, 3).

Table 3. The Distance Analysis of GP(8, 3)

Nodes in outer cycle	$v_{1,5}$	$v_{1,6}$	$v_{1,7}$	$v_{1,0}$	$v_{1,1}$	$v_{1,2}$	$v_{1,3}$	$v_{1,4}$
Distance of nodes in outer cycle	3	2	1	0	1	2	3	4
Distance of nodes in inner cycle	2	3	2	1	2	3	2	3
Nodes in inner cycle	$v_{0,7}$	$v_{0,2}$	$v_{0,5}$	$v_{0,0}$	$v_{0,3}$	$v_{0,6}$	$v_{0,1}$	$v_{0,4}$

In Table 4, when k = 2, the nodes with the reference addresses of 2 and 3 are adjacent to the nodes with reference addresses of 0 and 1 respectively. So the corresponding distances are respectively 1 and 2. Similarly, when k = 4, the nodes with reference addresses of -1, 0, 1, and 2 will be adjacent to the nodes of the corresponding reference addresses of 3, 4, 5, and 6.

Table 4. The Distance Table Starting With $v_{1,0}$

Reference Address of Outer Cycle	-1	0	1	2	3	4	5	6	7	8
Distances of Nodes in Outer Cycle (k = 4)	1	0	1	2	**3**	**3**	**4**	**5**	5	4

Distances of Nodes in Inner Cycle (k = 4)	2	1	2	3	**3**	**2**	**3**	**4**	4	3
Distances of Nodes in Inner Cycle (k = 3)	2	1	2	**3**	**2**	**3**	4	3	4	**5**
Distances of Nodes in Inner Cycle (k = 2)	**2**	1	2	**2**	**3**	3	4	**4**	**5**	5

In Table 4, if the number of nodes of the outer cycle in GP is 5, the 1st raw addresses of these nodes will be 3, 4, 0, 1, 2. If the number of nodes of the outer cycle in GP is 8, the 1st raw addresses of these nodes will be 5, 6, 7, 0, 1, 2, 3, 4.

LEMMA 3. In GP(m, k), $D(v_{i,0}, v_{i',j'}) = D(v_{i,0}, v_{i',m-j'}), j' \neq 0$.

Proof: According to the definition of GP(m, k), $v_{1,0}$ is adjacent to $v_{0,0}$. Since the nodes $v_{i,0}\, v_{i,1}, v_{i,2}, \ldots, v_{i,m-1}$ could be respectively addressed as $v_{i,m-1}, v_{i,m-2}, v_{i,m-3}, \ldots, v_{i,1}$, $D(v_{i,0}, v_{0,j'}) = D(v_{i,0}, v_{0,m-j'})$ and $D(v_{i,0}, v_{1,j'}) = D(v_{i,0}, v_{1,m-j'})$, therefore, $D(v_{i,0}, v_{i',j'}) = D(v_{i,0}, v_{i',m-j'})$.■

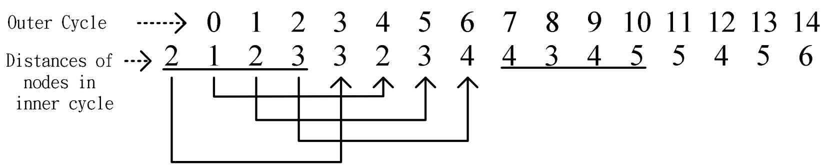


Figure 18. The Distance Value of Nodes of Inner and Outer Cycles in GP(m, 4).

In Figure 18, the distance values of the nodes in the right side of the source node $v_{1,0}$ can be affected merely by the upward and downward adjacent nodes. This is because the distance of right adjacent nodes is greater. Since we gradually approach the distance values to the adjacent nodes, we can get the distance value for each node.

In the discussion below, the symbol *CS* is an abbreviation of critical size and the symbol *shift* means the shift values. As a matter of convenience, we can adopt some specific nodes like $v_{0,0}$ and $v_{1,0}$ to make comparisons. In the inner cycle, the nodes will be linked with the interval of k nodes. If the 2nd coordinate of the address of a node in an outer cycle is no less than *CS*, the distance of this aforementioned outer cycle node will be 1 more than the distance of its adjacent inner cycle node. This mentioned property is detailed below.

LEMMA 4. In GP(m, k), assume $j \leq \lfloor m/2 \rfloor$, and $v_{1,j}$ is adjacent to $v_{0,j'}$. If $j \geq CS$, then $D(v_{1,0}, v_{1,j}) = D(v_{1,0}, v_{0,j'}) + 1$. Otherwise, $D(v_{1,0}, v_{1,j}) = j$, among them, when $k \leq 3$, $CS = 6$, and when $k > 3$, $CS = \lfloor (k+5)/2 \rfloor$.

Proof: If $v_{1,j}$ is not affected by the adjacent nodes of the inner cycle, then their distance of 0, 1, 2, and so on will gradually increase, such that $D(v_{1,0}, v_{1,j}) = j$. The maximum difference between the 2 adjacent nodes will be 1. As regards Figure 18, the first 2 sets of adjacent nodes in the inner cycles have been indicated with arrow marks to denote the influence of distance between the adjacent nodes.

We find that the distance of $v_{1,4}$ in Figure 18 should be changed to 3. For the specific node, $v_{1,CS}$, as the $v_{1,4}$ in Figure 18, $v_{1,CS}$ and its subsequent $v_{1,j}$ will satisfy $D(v_{1,0}, v_{1,j}) = D(v_{1,0}, v_{0,j'}) + 1$. Furthermore, according to Table 4, we discover that when $k \le 3$, $CS = 6$.

When $k > 3$, we adopt *shift* as a shift value. If the difference value is more than 2*shift*, then *CS* will change from *k* to *k* – *shift.* If *k* is an odd number, then the result will be $k - 3 - 2j_{shift} \ge 2$. If *k* is an even number, the result will be $k - 2 - 2shift \ge 2$. So, the maximum *shift* will be $\lfloor (k-5)/2 \rfloor$. Therefore, $CS = k - \lfloor (k-5)/2 \rfloor$, namely $\lfloor (k+5)/2 \rfloor$.■

From the above properties, we can deduce the distance values of the nodes in the outer cycle in GP. According to Lemmas 4 and 5, we can deduce the distance between any given node and the source node in GP.

LEMMA 5. In GP(m, k), assume $0 \le j \le \lceil m/2 \rceil$, $v_{1,j}$ is adjacent to $v_{0,j'}$, and $-\lfloor (k-1)/2 \rfloor \le shift \le \lfloor k/2 \rfloor$. If $j = k \times w + shift$, then $D(v_{1,0}, v_{0,j'}) = w + 1 + |shift|$.

Proof: According to the definition of GP(m, k), when $j = k \times w$ and $w > 0$, $v_{1,j}$ is adjacent to $v_{0,w}$. Furthermore, since $D(v_{1,0}, v_{1,j})$ is no less than w, $D(v_{1,0}, v_{0,w}) = w + 1$. When $j = k \times w \pm 1$, from Lemma 4, we find that $D(v_{1,0}, v_{0,j'}) = w + 1 + 1$. Similarly, when $j = k \times w + shift$ and $-\lfloor (k-1)/2 \rfloor \le shift \le \lfloor k/2 \rfloor$, the lemma is proved.■

2.6 ROUTING ALGORITHMS FOR GP

From the discussion of the properties of GP, we find that when the 2nd coordinate of the address of a node is equal to or greater than the critical size of GP(m, k), it should therefore route to another cycle's adjacent node. On the contrary, when the 2nd coordinate of the address of a node is less than its critical size, it should route to the nearest adjacent node of its own cycle. According to the previous discussion, we can deduce the shortest-path routing algorithm for GP. From the algorithm below, the 3rd row will first deduce the k_{iso} value discussed in Section 2.

Algorithm GP(m, k)_Routing(v_{i_s, j_s}, v_{i_d, j_d})
Begin
  $k_{iso} = k_{iso}$_of_GP(m, k)
  $j_{another}$ = GP(m, k)_ adjacent$_j$(v_{i_d, j_d})
  $v_{i,j} := v_{i_s, j_s}$ //{Set the source node.}
  while ($v_{i,j} \ne v_{i_d, j_d}$) **do**

```
//{The terminated condition of the loop is routing to the objective nodes.}
Begin
   if i = i_d then
   //{Node v_{i,j} and node v_{id,jd} is located within the identical cycle.}
      j_tmp := j_d          //{Set the value of j_tmp as j_d. }
   else
      j_tmp := j_another    //{Set the value of j_tmp as j_another. }
   end if
   if i = 1 then       //{v_{i,j} is located in the outer cycle. }
      if k ≤ 3 then    //{k value is equal to or less than 3.}
         CS := 6  //{To set the critical size as 6.}
      else
         CS := ⌊(k + 5)/2⌋  //{Set the critical size as ⌊(k + 5)/2⌋.}
      end if
   else  //{v_{i,j} is located in the inner cycle. }
      if k_iso ≤ 3 then
         CS := 6  //{Set the critical size as 6.}
      else
         CS := ⌊(k_iso + 5)/2⌋  //{Set the critical size as ⌊(k_iso + 5)/2⌋.}
      end if
   end if
   if (( j – j_tmp) mod m) ≥ CS then
   //{the distance between the address j of current node v_{i,j} and j_tmp is no less than
   //the critical size.}
      v_{i,j} : = v_{(i xor 1), j_another}       //{Move to the adjacent node in various cycles. }
   else
      if ((j – j_d + 1) mod m) < (( j – j_d) mod m) then
         v_{i,j} := v_{i, (j + 1) mod m}   //{Move to a adjacent node in the same cycle. }
      else
         v_{i,j} := v_{i, (j – 1) mod m}   //{Move to a adjacent node in the same cycle. }
      end if
   end if
end while
end Algorithm
```

The function GP(m, k)_adjacent$_j$($v_{i,j}$) is based on the definition of GP(m, k) and returns to the 2[nd] coordinate of the address of the adjacent node located on another cycle.

```
function GP(m, k)_adjacent_j(v_{i,j})
begin
  if i = 1 then   //{v_{i,j} is located on the outer cycle.}
     for w := 0 to m – 1 do
     begin
        if ((m×w + j) mod k) = 0 then
        //{According to the definition 2, we can determine whether it is the 2nd
        //coordinate value on the adjacent nodes located on another cycle. }
           return (m×w + j)/k
```

```
      end if
    end for
  else //{v_{i,j} is located on the inner cycle.}
    for w := 0 to m – 1 do
    begin
      if ((m×w + j) mod k_iso_of_GP(m, k)) = 0 then
        return (m×w + j) / k_iso_of_GP(m, k)
      end if
    end for
  end if
end function
```

The function k_{iso}_of_GP(m, k) is based on the Lemma 2 discussed above. The result is the k value after the exchange between the outer and inner cycles.

```
function k_iso_of_GP(m, k)
begin
  for w := 0 to m – 1 do
  begin
    if ((m×w + 1) mod k) = 0 then
      k_tmp := (m×w + 1)/k
    end if
  end for
  return min(k_tmp , m – k_tmp)
end function
```

3 Novel Y Graphs

Graphs for interconnected networks [5] are a widely discussed subject in parallel and distributed systems. The subject proposed in this section is the novel Y graph with GP as its special case.

3.1 Definition

All the nodes in a bipartite graph could be divided into 2 groups to ensure that any two nodes in either group are not adjacent nodes. In Figure 19, the hypercube of n dimensions can be designated as Q_n and are bipartite graphs. In Figure 20, Torus graphs of n dimensions can be designated as T(m_0, m_1, .., m_{n-1}). Among them, m_0, m_1 .., m_{n-1} signifies the size of each dimension. When m_0, m_1 .., m_{n-1} are all even numbers, T(m_0, m_1, .., m_{n-1}) is a bipartite graph. The connected graphs indicate that there are paths between any two given nodes.

As the hypercube and folded Petersen expansion demonstrate, most interconnected networks can be viewed as the product expansion of graphs. However, the novel graphs proposed in this paper involve a new and efficient expansion method of interconnecting networks. The advantages of Y graphs are regularity [6] and short diameters [7] with the definition described below.

DEFINITION 3. G is a connecting and bipartite graph. $Y_0(G)_{m,\,k} = G$ and $Y_1(G)_{m,\,k}$ are defined as follows: All the nodes of G will be categorized as two groups of $v_{i,j} \in V_0$ and $v_{i',j'} \in V_1$., and ensure that any two given nodes in either group are not adjacent nodes. If $v_{i,j}$ is adjacent to $v_{i',j'}$, then two C_m cycles respectively will replace $v_{i,j}$ and $v_{i',j'}$ and will connect them just like two C_m cycles in GP(m, k). We define $Y_n(G)_{m,\,k} = Y_1(Y_{n-1}(G)_{m,\,k})_{m,\,k}$ with the number of dimensions $n > 0$. We define $Y_{-n}(G)_{m,\,k}$ as a $Y_n(G)_{m,\,k}$ graph whose every node is replaced with C_{m_deg} and which makes each edge in the original node link to a new node in C_{m_deg} according to the order of its dimensions. Among them, m_deg is the degree of the original node.

3.2 TOPOLOGICAL PROPERTIES

LEMMA 6. $Y_1(Q_1)_{m,\,k}$ and GP(m, k) are isomorphic graphs of each other. Assume $n > 0$. There are $2^{n'}m^n$ nodes in $Y_n(Q_{n'})_{m,\,k}$ and each node is with degrees of $n' + 2n$. There are $2^{n'}m^n(n' + 2n)$ nodes in $Y_{-n}(Q_{n'})_{m,\,k}$ and each node is with degrees of 3.

Proof: According to definition of the $Y_n(G)_{m,\,k}$, we find that $Y_1(Q_1)_{m,\,k}$ is the regular graph with its degree of 3 and it is isomorphic to GP(m, k).

According to definition 3, in $Y_n(Q_{n'})_{m,\,k}$, whenever n is added with 1, the degree of each node in the graph will be increased with 2. Because each node will be replaced with C_m, the number of nodes will be m times as before. Since the degrees of each node in $Y_0(Q_{n'})_{m,\,k}$ are equal to the degrees of each node in $Q_{n'}$ with the value of n', and the number of nodes in $Y_0(Q_{n'})_{m,\,k}$ is $2^{n'}$, the degrees of $Y_n(Q_{n'})_{m,\,k}$ are $n' + 2n$ and the number of nodes in $Y_n(Q_{n'})_{m,\,k}$ is $2^{n'}m^n$.

Furthermore, when $n > 0$, the number of nodes in $Y_{-n}(Q_{n'})_{m,\,k}$ will be identical to the number of nodes in $Y_n(Q_{n'})_{m,\,k}$ multiplying the degrees of each node in $Y_n(Q_{n'})_{m,\,k}$, namely $2^{n'}m^n(n' + 2n)$; in each node, since there are two edges to link to the nodes of the cycle and an edge to link another node, the degrees of each node in $Y_{-n}(Q_{n'})_{m,\,k}$ are 3.■

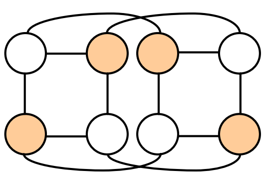


Figure 19. A bipartite graph, Q_3.

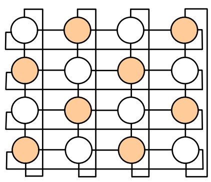


Figure 20. A bipartite graph, T(4, 4).

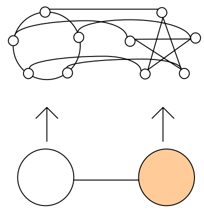


Figure 21. $Y_1(Q_1)_{5,2}$.

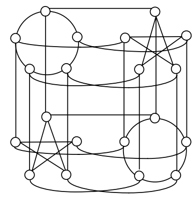


Figure 22. $Y_1(Q_2)_{5,2}$.

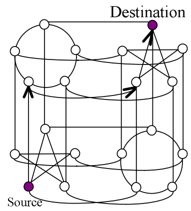


Figure 23. A Routing example.

LEMMA 7. The diameter of $Y_n(G)_{m,k}$ is Dia(G) + n × (Dia(GP(m, k)) – 1).

Proof: Dia($Y_1(G)_{m,k}$) = Dia(G) – 1 + Dia(GP(m, k)), since we could start routing from the source node to any node in the cycles adjacent to the cycle of the destination node, and continue to route to the destination node according to the routing algorithms of GP, shown as Figure 23. Furthermore, $Y_n(G)_{m,k} = Y_1(Y_{n-1}(G)_{m,k})_{m,k}$, so Dia($Y_n(G)_{m,k}$) = Dia($Y_{n-1}(G)_{m,k})_{m,k}$) – 1 + Dia(GP($m$, k)) = Dia($Y_{n-2}(G)_{m,k})_{m,k}$) – 2 + 2Dia(GP(m, k)) = Dia($Y_0(G)_{m,k}$) – n + n×Dia(GP(m, k)) and according to definition 3, $Y_0(G)_{m,k} = G$. Therefore, the lemma is proved. ■

3.3 FAULT TOLERANCE

This section will analyze the fault tolerance of the Y graph, and make a comparison with the widely applied hypercube graph. We adopt F(G) to indicate the possibility that graph G might not be able to connect to the other nodes. Since the number of nodes in Q_n is 2^n, the fault will only exist if n pieces of adjacent nodes in 1 specific node are all found to be at fault. So

$$F(Q_n) = 2^n/C(2^n, n)$$

Among them, the denominator C(2^n, n) indicates the means available to choose n nodes from 2^n nodes.

When considering $Y_1(Q_{n-2})_{5,2}$, since it comes in $5\times2^{n-2} = 1.25\times2^n$ nodes and each node comes in n pieces of adjacent nodes, then

$$F(Y_1(Q_{n-2})_{5,2}) = 1.25\times2^n / C(1.25\times2^n, n)$$

To compare both possibilities,

$$F(Q_n) / F(Y_1(Q_{n-2})_{5,2}) = 1.25\times C(2^n, n) / C(1.25\times2^n, n) << 1$$

This shows that the fault tolerance of $Y_1(Q_{n-2})_{5,2}$ is far higher than that of the hypercube. Consequently, if graph Y acts as the base graph for parallel computers, high fault tolerance will be one of their outstanding features.

3.4 ROUTING ALGORITHMS

The routing algorithm of graph Y is well simplified because of the routing algorithm of GP(m, k) proposed in Section 2. We adopt V_s and V_d to respectively indicate the source nodes and destination nodes. The whole process of routing algorithms could be divided as 2 procedures of global routing and local routing, as described below.

Algorithm $Y_n(G)_{m,k}$_Routing(V_s , V_d)

Step1: By using the routing method of graph G, to route messages from the source node V_s to any node in the cycles adjacent to the cycle that V_d is in. **(Global routing)**

//{Only one step of routing is needed to reach the next cycle according to the definition of $Y_n(G)_{m,k}$. The number of cycles adjacent to the cycle that V_d is in could be more than 1, and we could then select any one cycle from them. }

Step2: By repeatedly using the routing method of graph GP(m, k), to route messages from the current node to the destination node. **(Local routing)**

The complexity of the routing algorithm is the size of distance between the source and destination nodes, and the diameter of the Y graph has been described in Lemma 7. The routing of GP has been discussed in Section 2. The reference graph G is any given bipartite graph, for example, hypercube, tree and so on.

Furthermore, by definition 3, if GP(m', k') is not a bipartite graph, then there's no $Y_n(G)_{m',k'}$ where $|n| > 1$.

Table 5. Comparison between the Novel Graph Y and Others

	Nodes	Degree	Diameter	Note
Hypercube, Q_n	2^n	n	n	Product Network
$Y_1(Q_{n-3})_{8,3}$	2^n	$n-1$	n	
$Y_1(Q_{n-2})_{5,2}$	1.25×2^n	n	$n-1$	
Hyper Petersen, HP_n	1.25×2^n	n	$n-1$	Product Network , GP(5, 2)×Q_{n-3}
Folded Petersen, FP_k	10^k	$3k$	$2k$	Product Network
Folded Petersen Cube, $FPQ_{n,k}$	$2^n\times 10^k$	$n+3k$	$n+2k$	Product Network
Folded n-cube	2^n	$n+1$	$\lceil n/2 \rceil$	Product Network
k-ary n-cube	k^n	$2n$	$n\times\lfloor k/2 \rfloor$	Product Network
Cyclic-Cubes, G_n^k	$n\times k^n$	$2k$	$\lfloor 3n/2 \rfloor$	Product Network
CCC	$n\times 2^n$	3	$O(n)$	
$Y_{-1}(Q_{n-3})_{8,3}$	$2^n(n-1)$	3	$O(n)$	
$Y_{-1}(Q_{n-2})_{5,2}$	$1.25n\times 2^n$	3	$O(n)$	
$Y_n(Q_{n'})_{8,3}$	$2^{n'+3n}$	$n'+2n$	$n'+3n$	

3.5 Comparison

This section is focusing on the comparison between the graphs we proposed previously and other graphs. After the comparison of these graphs, we found that the new graph Y proposed by us will be able to reach an identical effect or even better.

When compared with the popular hypercube, the $Y_1(Q_{n-2})_{5,2}$ graph not only contains up to 25% more nodes, but also has smaller diameter. $Y_1(Q_{n-3})_{8,3}$ has smaller degrees than the hypercube, therefore it has a smaller cost. And we can see that the novel graph Y of n dimensions, for example, $Y_n(Q_{n'})_{8,3}$ in Table 5, have high performance and low relative cost. The comparison results are presented as Table 5.

4 EMBEDDING AND VLSI LAYOUT

4.1 EMBEDDING

The embedding [9] and fault tolerance are correlated closely and they are one of the most frequently discussed subjects in the investigation of parallel and distributed systems.

A_m is defined as the linear array with m nodes. If the graph is embedded by A_m, then it shows that this graph comes in the Hamilton path with the length of m.

LEMMA 8. A_{2m} is available to embed in $Y_1(Q_1)_{m,k}$.

Proof: According to Lemma 6, $Y_1(Q_1)_{m,k}$ and GP(m, k) are isomorphic graphs since the 2nd coordinate of address of nodes in the inner or outer cycles are respectively 0, 1, 2, …, $m-1$ and adjacent to each other in this order. In addition, $v_{1,0}$ is adjacent to $v_{0,0}$. From the connecting order of $v_{1,1}$, $v_{1,2}$, …, $v_{1,0}$, $v_{0,0}$, $v_{0,1}$, …, $v_{0,m-1}$ we can obtain A_{2m}.■

LEMMA 9. The number of G subgraphs in $Y_1(G)_{m,k}$ is m.

Proof: According to definition 3 and referring to Figure 22, the nodes in the reference graph G are replaced with the cycle C_m, and constructing the graph Y does not involve the interconnection of the reference graph G. Therefore, the number of G subgraphs in $Y_1(G)_{m,k}$ is equal to the number of nodes in C_m, namely m.■

LEMMA 10. Q_n is available to embed in $Y_1(Q_n)_{m,k}$.

Proof: According to Lemma 9, $Y_1(Q_n)_{m,k}$ comes in m Q_n subgraphs. Therefore, $Y_1(Q_n)_{m,k}$ is readily embedded by Q_n.■

LEMMA 11. $Y_1(Q_n)_{m,k}$ can be embedded by a complete binary tree with the number of nodes up to 2^n-1.

Proof: The property of the hypercube has been discussed in detail and it is known that the hypercube is capable of being embedded with the complete binary tree that has $2^{n-1}-1$ nodes. In $Y_1(Q_n)_{m,k}$, we can select any 3 adjacent nodes in a cycle that replace the node of G. They are referred to as left, middle and right node, in that order. According to Lemma 10, there is one Q_n that contains a left node, and another Q_n that contains a right node. Therefore, two complete binary trees exist, respectively rooted as left node and right node, both with the number of nodes up to $2^{n-1} - 1$. In addition, the above-mentioned middle node is adjacent to the left and right nodes. Therefore, the number of nodes in the complete binary tree is up to $1 + (2^{n-1} - 1) + (2^{n-1} - 1) = 2^n - 1$.■

4.2 VLSI LAYOUT

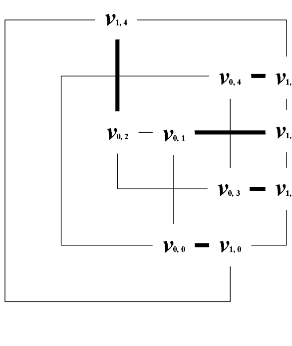
In chip design, the VLSI layout [1] is an important procedure that affects the cost and performance of chips. After investigating the GP, we continue to discuss the generalized method of VLSI layout [11]. In Figure 24, the outer cycle is designated as the outmost circuit with the outer cycle node located on. This means that the spiral circuit of the inner cycle will be wound with k rounds. In Figures 24 and 25, k = 2, and there are 2 rounds of spiral circuits on the inner cycle. In Figure 26, k = 3 and there are 3 rounds of spiral circuits on the inner cycle.

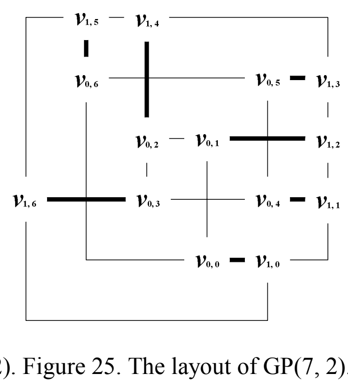
Referring to Figure 9, since the interconnection between the nodes in GP(8, 3) corresponds to Figure 26, we can place in the inner cycle nodes of 3, 4, 5 as shown in Figure 9, when it is wound the 2nd round. So the middle position of the layout is located on the 2nd round. The whole layout method applicable to GP is described below.

Step1: Place the circuit of outer cycles and indicate the spiral circuit of inner cycles.

Step2: Establish the interconnecting data shown as Figure 9.

Step3: Place and connect the nodes of $v_{1,0}$ and $v_{0,0}$, shown as Figure 26. In the order of the outer cycle addressed with 1, 2… m – 1, place and connect the nodes located on inner and outer cycles. Among them, the location of inner cycle nodes will refer to the interconnecting relation, according to the row number of this mentioned node, so as to determine the location in spiral circuits.





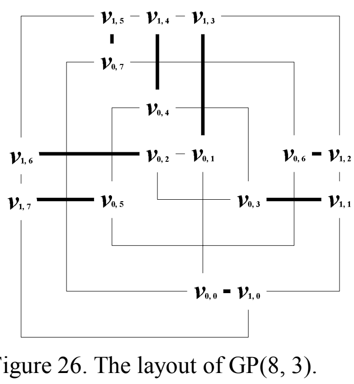


Figure 24. The layout of GP(5, 2). Figure 25. The layout of GP(7, 2). Figure 26. The layout of GP(8, 3).

5 CONCLUSIONS

GP comes in a wide range of nodes. We have proven that there exists GP with nodes of even numbers more than 12, and the degree will be fixed to 3. We first proposed the relevant properties to discuss, like symmetry and distance…etc. GP are, as Petersen graph, applicable to the relevant expansion.

The problem of routing in GP is solved for the first time. Also, in Section 2, we proposed the relevant theorems and properties so as to deduce the distance between any given 2 nodes of GP, together with the diameters. This paper also includes the VLSI-layout method and shortest-path routing algorithm of graph Y.

This study has covered the investigation for symmetric graphs with degree of 3. We have found the investigation results of their properties and the shortest paths, together with the infinite number of nodes. The new graph Y proposed in this paper can create novel expansions of interconnection. We can expand our investigation in the application of computing networks, self-organization, and artificial intelligence [15].

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