Feedback Vertex Sets in Star Graphs

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Abstract

In a graph $G = (V, E)$, a subset $F(G)$ of $V(G)$ is a feedback vertex set if the subgraph induced by $V(G) \setminus F(G)$ is acyclic. In this paper, we find a lower bound and an upper bound to the size of the feedback vertex set for star graphs.

Keyword: Feedback vertex set, Interconnection network, Star graphs.

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1 Introduction

Let $G = (V, E)$ be a simple graph, i.e., loopless and without multiple edges, with vertex set $V(G)$ and edge set $E(G)$. A set of vertices $F(G) \subseteq V(G)$ is called a *feedback vertex set* if the subgraph induced by $V(G) \setminus F(G)$ is acyclic, where $V(G) \setminus F(G) = \{x | x \in V(G)$ and $x \notin F(G)\}$. If the cardinality of $F(G)$, denoted by $|F(G)|$, is minimum among all possible feedback vertex sets, then we call it a *minimum feedback vertex set*.

The problem of finding a minimum feedback vertex set is NP-hard for general graphs [9]. The approximation ratio of the best known approximation algorithm for this problem is 2 [4]. Besides, most of the work have been devoted to solving the problem for certain classes of graphs. Polynomial time algorithms have been found for some special graphs, e.g., reducible graphs[15], cocomparaibility[10], convex bipartite graphs[10], cyclically reducible graphs[16], and interval graphs[12]. On meshes, toris, butterflies, cube connected cycles and hypercubes, the lower and upper bounds to the size of the feedback vertex set are established and improved recently [5, 7, 13].

The problem has important applications to several fields. For example, we consider the deadlock prevention in operating systems. Once a deadlock has been detected, some strategy is needed to recovery. Usually, a deadlock in a system can be described by using a wait-for graph [14]. In a wait-for graph, each vertex represents a process, and the existence of an edge (i, j) indicates that process i is waiting for process j to release a resource requested by process i. A deadlock exists in a system if and only if the corresponding wait-for graph contains a cycle. One of the best-known approach for solving the deadlock problem can be carried out to abort some deadlocked processes in the wait-for graph. Using graph-theoretic terminology, the strategy is equivalent to finding a (minimum) feedback vertex set for such a system.

We consider the problem for a particular interconnection network, namely, star graphs.

Star graphs were proposed as an attractive alternative to hypercubes with many nice topological properties [1, 2]. Both star graphs and hypercubes provide attractive interconnection scheme for massively parallel systems. Hence characterizations of these architectures have been widely investigated. The star graphs are vertex and edge symmetric, highly regular, strongly hierarchical, and maximally fault-tolerant. Due to the strongly hierarchical structure, a star graph can be defined recursively. Moreover, star graphs have many superior advantages over hypercubes such as smaller degree and diameter. In this paper, we give a lower bound and an upper bound to the size of the minimum feedback vertex for star graphs.

The remaining part of this paper is organized as follows. In next section, we present a simple algorithm for finding an upper bound to the size of the minimum feedback vertex on star graphs. In contrast, we also give a lower bound to the problem on regular graphs. Finally, some concluding remarks are given in the last section.

2 Main Results

The *n-dimensional star graph* (n-star for short), denoted by S_n, is an undirected graph consisting of $n!$ vertices labeled with distinct permutations $[p_1, p_2, \ldots, p_n]$ from the set of symbols $N = \{1, 2, \ldots, n\}$. Two vertices are connected by an edge if and only if the label of one can be obtained from the label of the other by swapping the first symbol (conventionally, the leftmost) and the ith symbol, where $2 \leq i \leq n$ [1, 2]. Figure 1 depicts S_4 which contains 24 vertices, where symbols a, b, c and d indicate the connection through the same symbol. Vertices $[1, 2, 3, 4]$ and $[4, 2, 3, 1]$ are neighbors since their labels differ only in the first and the last positions and swapping these two symbols of one vertex becomes the label of another vertex. Note that an n-star is an edge- and vertex-symmetric regular graph of degree $n - 1$.

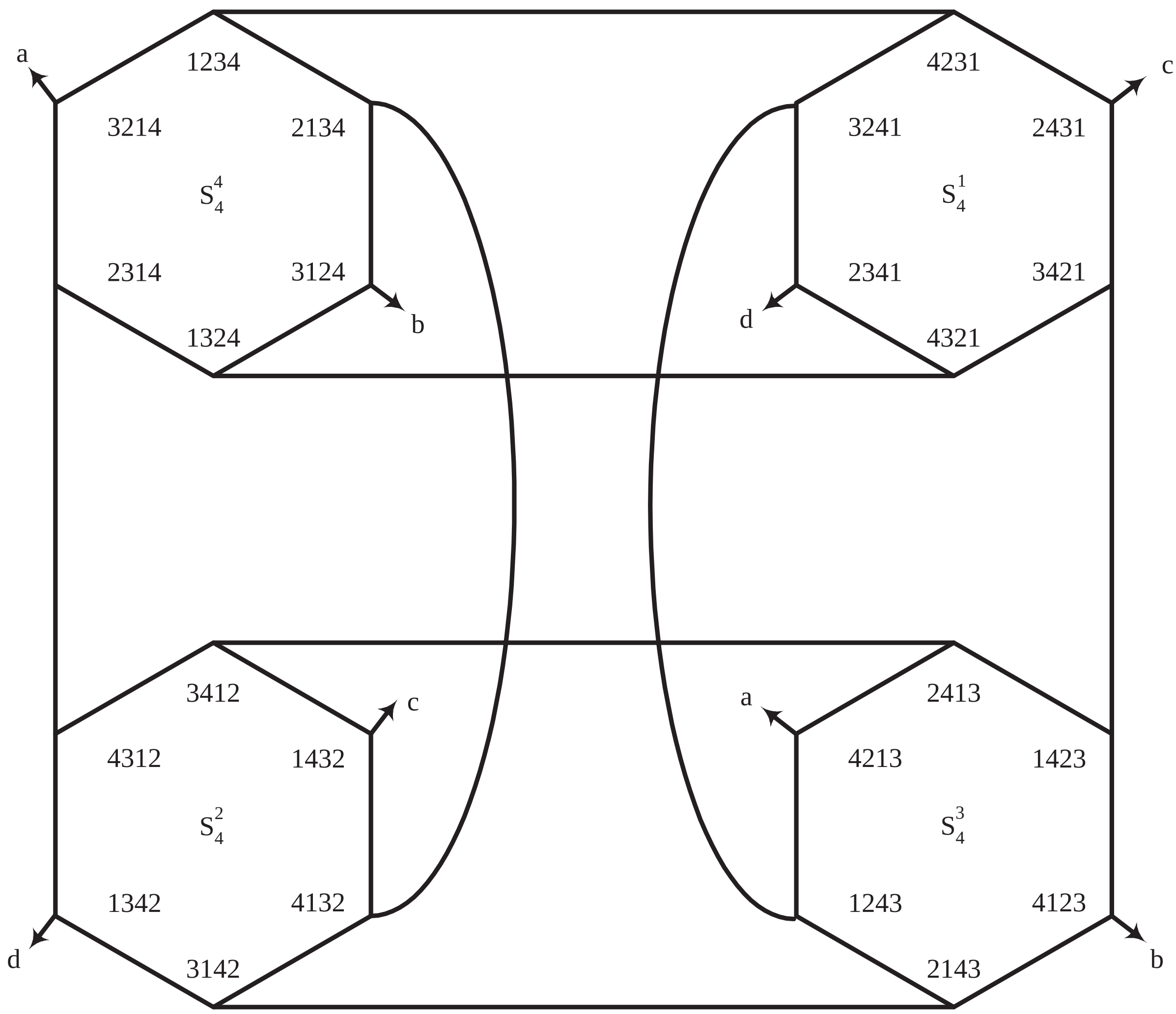

Figure 1: 4-dimensional star graph S_4.

Let $[1, 2, \ldots, n]$ denote the *identity permutation* in S_n. A permutation is odd (resp. even) if it can be turned to the identity permutation through odd (resp. even) number of transitions. Since the star graph is a bipartite graph with equal partite size, half of the vertices are in one partite set [?]. An *independent set* S of a graph G is a set of vertices such that no two vertices of S are adjacent in G. If the cardinality of S is the maximum among all possible independent sets, the set is called a *maximum independent set*. Let I be the set of vertices with even permutations, then I is a maximum independent set of S_n, where $|I| = \frac{n!}{2}$. Furthermore, the subgraph induced by I, denoted $< I >$, has no cycles in S_n. So $V(S_n) \setminus I$ is a trivial feedback vertex set.

For $i, j \in N$, let $N_i = N \setminus \{i\}$ and $N_{i,j} = N \setminus \{i, i+1, \ldots, j\}$, where $i < j$. Define two classes of sets as follows.

$$\Phi_1 = \{[1, p_2, p_3, \ldots, p_n] \,|\, p_2, p_3, \ldots, p_n \in N_1 \text{ and } p_j \neq p_k \text{ if } j \neq k\}, \text{ and}$$
$$\Phi_i = \{[i, p_2, p_3, \ldots, p_{n-i+1}, i-1, i-2, \ldots, 2, 1] \,|\, p_2, p_3, \ldots, p_{n-i+1} \in N_{1,i} \text{ and } p_j \neq p_k \text{ if } j \neq k\}, \text{ for } 2 \leq i \leq n.$$

It is obvious that $\Phi_i, 1 \leq i \leq n-1$, are all independents for S_n. Let $G_i, 1 \leq i \leq n-1$, be the subgraph induced by $I \bigcup \Phi_1 \bigcup \Phi_2 \bigcup \ldots \bigcup \Phi_i$. We shall show that $V(G) \setminus V(G_i), 1 \leq i \leq n-1$, is a feedback vertex set and it is smaller than $V(S_n) \setminus I$.

The neighborhood $N(v)$ of a vertex v is the set of vertices which are adjacent with v. A vertex $v \in G_i, 1 \leq i \leq n-1$, is called a *port vertex* of G_i if there exists a vertex $u \in \Phi_j$ and $j > i$ such that $u \in N(v)$. We use Figure 2 as an example to illustrate the above notation. Figure 2(a) depicts the induced subgraph $< I >$ of S_4. And independent sets $\Phi_1 = \{[1, 3, 2, 4], [1, 2, 4, 3], [1, 4, 3, 2]\}, \Phi_2 = \{[2, 3, 4, 1]\}$ and $\Phi_3 = \{[3, 4, 2, 1]\}$ are sketched in Figure 2(b),(c), and (d), respectively. Consider the induced subgraph G_1 of S_4 in Figure 2(b). Vertices $[4, 3, 2, 1], [3, 2, 4, 1], [2, 4, 3, 1], [1, 3, 4, 2]$ and $[1, 4, 2, 3]$ are port vertices of G_1, since vertices $[4, 3, 2, 1], [1, 3, 4, 2]$ and $[3, 2, 4, 1]$ are neighbors of the

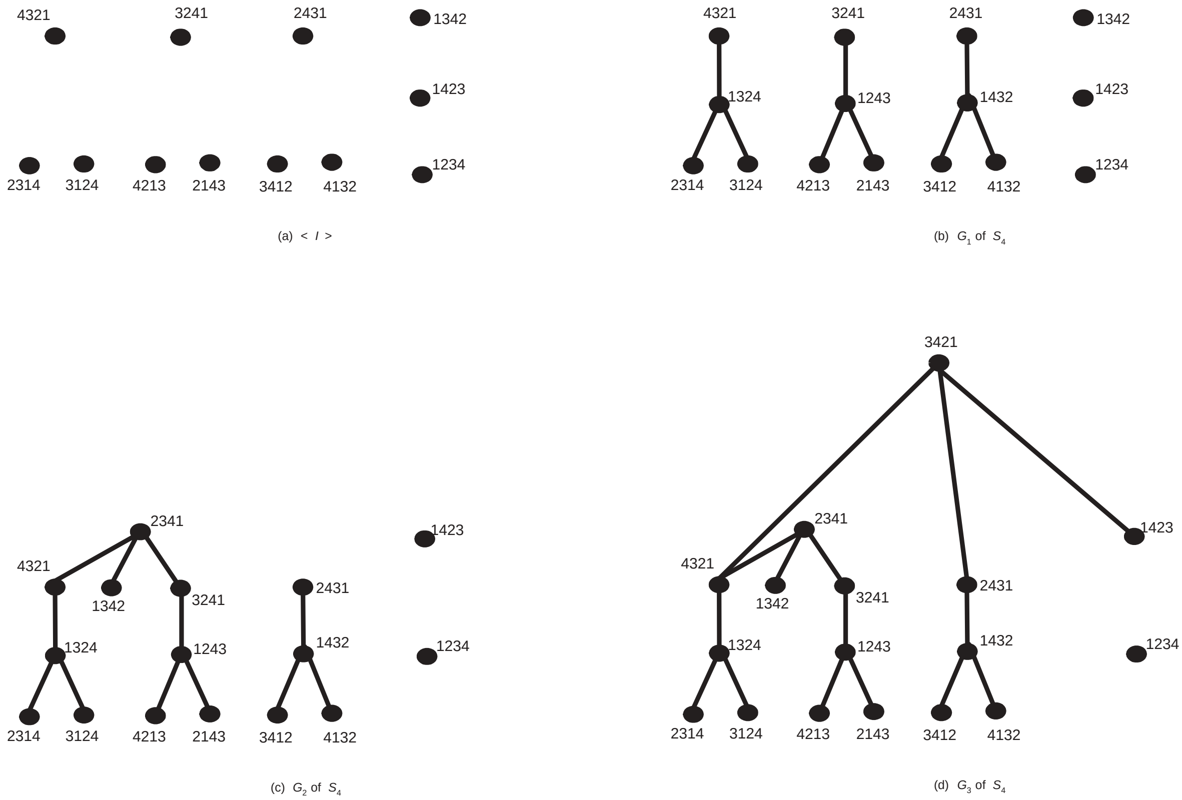

Figure 2: Acyclic subgraph $< I >, G_1, G_2, G_3$ of S_4, respectively in figures (a), (b), (c), and (d).

vertex $[2,3,4,1] \in \Phi_2$, and vertices $[2,4,3,1]$ and $[1,4,2,3]$ are neighbors of the vertex $[3,4,2,1] \in \Phi_3$. Figure 2(c) illustrates G_2 of S_4 where vertices $[4,3,2,1], [2,4,3,1]$ and $[1,4,2,3]$ are port vertices of G_2. But the vertices $[1,3,4,2]$ and $[3,2,4,1]$ are not port vertices of G_2, since they are not adjacent with any vertex in Φ_3. Furthermore, Figure 2(d) is a maximum acyclic induced subgraph of S_4. Thus, $V(S_4) \setminus V(G_3)$ is a minimum feedback vertex set.

A set $D \subseteq V(G)$ is a *dominating set* of G if for every vertex $u \in V(G) \setminus D$ there exists a vertex $v \in D$ such that u is adjacent to v. In particular, we call D a *perfect dominating set* if every vertex in $V(G) \setminus D$ is adjacent to exactly one vertex in D. We call D an *independent dominating set* if D is also an independent set of G. A dominating set D is *independent perfect* if it is both independent and perfect. In star graphs, Arumugam and Kala [3] showed that Φ_1 is not only a minimum independent dominating set, but also a minimum perfect dominating set. That is to say, Φ_1 is in fact a minimum independent perfect dominating set.

Lemma 1 *G_1 is acyclic and each component of G_1 has at most one port vertex.*

Proof. Let u, v be two vertices of G_1. Since Φ_1 is a minimum independent perfect dominating set, there is no common vertex between the neighbors of u and v. Since u and v are not adjacent and the neighbors of u and v belong to the independent set I, u and v are in different components of G_1. Since each vertex of I is adjacent with at most one vertex of Φ_1, the component of G_1 is either an isolated vertex or a nontrivial tree. Thus, G_1 is acyclic. To complete the proof, let T be a component of G_1. We now consider two cases, depending on T is an isolated vertex or a nontrivial tree. Let T be an isolated vertex. It is clear that T is the only possible vertex which is adjacent with some vertex in $\Phi_i, i > 1$. Thus, the component T has one port vertex if and only if T is a port vertex. Otherwise, T is a nontrivial tree. Let $r \in \Phi_1$ be the root of T. Since r is a odd permutation vertex, the $n-1$ neighbors of r are even permutation vertices of I. Since the first (leftmost) symbol of r is one. Thus, T has exactly one leaf whose last (rightmost) symbol is one. For the other leaves, since both the first and the last symbols are not one, they are adjacent with none of the vertex in $\Phi_i, i \geq 2$. So, the vertex in each tree with the last symbol as one is the only possible port vertex.

Q. E. D.

For each vertex $v = [p_1, p_2, \ldots, p_n]$ of S_n, let $NB_i(v), 2 \leq i \leq n$, be the i-th neighbor of v. That is $NB_i(v) = [p_i, p_2, \ldots, p_{i-1}, p_1, p_{i+1}, \ldots, p_n]$.

Lemma 2 *Each component of $G_k, 1 \leq k \leq n-1$, has at most one port vertex.*

Proof. We proceed by induction on k. For $k = 1$, the lemma holds by Lemma 1. As the inductive hypothesis we assume that each component of $G_k, 2 \leq k \leq i-1$, has at most one port vertex. We now want to show that the lemma is also true for $k = i$. Let u, v be two distinct vertices in Φ_i. Since Φ_i is an independent set, u and v are not adjacent. Let T be a component of G_{i-1}. By hypothesis, T has at most one port vertex. Suppose that T has no port vertex, it is easy to see that T still has no port vertex in G_i. Consequently, we assume that T has exactly one port vertex. Let $T(p)$ be the port vertex of T. If $T(p)$ is not adjacent to any vertex of Φ_i, then $T(p)$ is still the unique port vertex of T in G_i. Thus, we assume that $T(p)$ is a neighbor of a vertex $w \in \Phi_i$. Let $w = [i, p_2, \ldots, p_{n-i+1}, i-1, i-2, \ldots, 1]$,$w \in \Phi_i$. To complete the proof, we shall show that there are at most one of the neighbors of w which is a port vertex in G_i . Consider the neighbors $NB_j(w), 2 \leq j \leq n$, of vertex w. Suppose that $NB_j(w)$ is a port vertex in G_i. Then there exists a vertex $\phi = [i', p_2, \ldots, p_{n-i}, i, i-1, \ldots, 1]$ in $G_{i'}, i' > i$, such that $NB_j(w) \in N(\phi)$. There are now four cases to consider, depending on the number j.

Case 1. $2 \leq j \leq n-i$. Let $NB_j(w) = [p_j, p_2, \ldots, p_{j-1}, i, p_{j+1}, \ldots, p_{n-i+1}, i-1, i-2, \ldots, 1]$. Compare the positions of symbol i in $v \in N(\phi)$ and $NB_j(w)$. Since the symbol i of v occurs either in the first or the (n-i+1)-th position, which is different from the position of $NB_j(w)$. Thus, $NB_j(w)$ and ϕ are not adjacent. It contradicts that $NB_j(w) \in N(\phi)$.

Case 2. $n-i+2 \leq j \leq n-1$. Let $NB_j(w) = [p_j, p_2, \ldots, p_{n-i+1}, i-1, i-2, \ldots, p_{j-1}, i, p_{j+1}, \ldots, 1]$. The proof is similar to Case (1).

Case 3. $j = n$. Let $NB_n(w) = [1, p_2, \ldots, p_{n-i+1}, i-1, i-2, \ldots, 2, i]$. Since the symbols in the first and the n-th positions of ϕ are i' and 1, respectively. If $NB_n(w) \in N(\phi)$ then $NB_n(w) = NB_n(\phi)$, which implies that $i = i'$. It is a contradiction.

Case 4. $j = n-i+1$. If $p_{n-i+1} = i+1$ then there exists a vertex $x \in \Phi_{i+1}$ such that $NB_{n-i+1}(w)$ and x are adjacent. Thus, $NB_{n-i+1}(w)$ is a port vertex of T in G_i.

In accordance with the above discussion, $NB_j(w)$ is a port vertex only if $j = n-i+$

1(Case 4).

Q. E. D.

Lemma 3 *G_i is acyclic, for $1 \le i \le n$.*

Proof. The proof is also by induction on i. For $i = 1$, the lemma holds by Lemma 1. Assume the theorem to be true for all $G_i, i < n-1$. In G_{n-1}, by Lemma 2, since each component of G_{i-1} is acyclic and has at most one port vertex, each acyclic component is incident with at most one vertex in Φ_{n-1}. Because Φ_{n-1} is an independent set, the component in G_{n-1} is also acyclic, completing the proof.

Q. E. D.

Theorem 4 $|F(S_n)| \le \frac{1}{2}[n! - (n-1)! - (n-2)! - \ldots - 2!] - 1.$

Proof. By Lemma 3, G_{n-1} is acyclic. Thus, $V(G) \setminus V(G_n - 1)$ is a feedback vertex set. Therefore,

$$\begin{aligned} |F(S_n)| &\le |V(G) \setminus V(G_{n-1})| \\ &= n! - [|I| + |\Phi_1| + |\Phi_2| + \ldots + |\Phi_{n-1}|] \\ &= n! - [\tfrac{n!}{2} + \tfrac{(n-1)!}{2} + \ldots + \tfrac{2!}{2} + 1] \\ &= \tfrac{1}{2}[n! - (n-1)! - (n-2)! - \ldots - 2!] - 1. \end{aligned}$$

Q. E. D.

To find the lower bound to the size of the feedback vertex set for star graphs, an analysis based on [7] is given. In [7], Focardi and Luccio state a lower bound related to the number of components of the resulting acyclic subgraph for hypercubes. We now establish an equation, a modification of [7], for calculating the lower bound to the size of the feedback vertex set in $r-$regular graphs.

Theorem 5 *Given an $r-$regular graph $G = (V, E)$, $|F(G)| \geq |V(G)| - \frac{|E(G)|-rc}{r-2} - c$*

Proof. Let $G' = (V - F(G), E')$ be the acyclic induced subgraph of G. Let c be the number of components in G' and T_i be the i-th component. The degree of v in T_i is denoted by $d^{in}(v)$. For simplicity, let $V_i, E_i, E_i^{in}, E_i^{out}$ denote the number of vertices, edges, internal edges, and external edges of T_i, respectively. Since T_i is a tree, $|V_i| = |E_i^{in}| + 1$. To count the external edges of T_i, we have

$$
\begin{aligned}
|E_i^{out}| &= \Sigma_{v \in V_i}(r - d^{in}(v)) \\
&= r(|V_i|) - \Sigma_{v \in V_i}(d^{in}(v)) \\
&= r(|V_i|) - 2(|V_i| - 1) \\
&= (r-2)(|V_i|) + 2 \\
&= (r-2)(|E_i| + 1) + 2 \\
&= (r-2)|E_i| + r.
\end{aligned}
$$

Then, the cardinality of the external edges in G is

$$
\begin{aligned}
|E(G) - E'(G')| &= \Sigma_{i=1}^{c}|E_i^{out}| \\
&= (r-2)|E'(G')| + rc.
\end{aligned}
$$

Since

$$
\begin{aligned}
|E(G)| &\geq |E'(G')| + |E^{out}| \\
&= |E'(G')| + (r-2)|E'(G')| + rc \\
&= (r-1)|E'(G')| + rc,
\end{aligned}
$$

we have $|E'(G')| \leq \frac{|E(G)|-rc}{r-1}$.

The size of the feedback vertex set is

$$
\begin{aligned}
|F(G)| &= |V(G)| - \Sigma_{i=1}^{c}|V_i| \\
&= |V(G)| - \Sigma_{i=1}^{c}|E_i + 1|
\end{aligned}
$$

$= |V(G)| - |E'(G')| - c.$

So,

$|F(G)| \geq |V(G)| - \frac{|E(G)| - rc}{r-1} - c.$

Q. E. D.

Note that the n-star has $n!$ vertices, $\frac{n!(n-1)}{2}$ edges and degree $n-1$. The next result follows.

Corollary 6 $|F(S_n)| \geq \frac{(n-3)n!+2}{2(n-2)}, n \geq 3$

By Theorem 4 and Corollary 6, we get $\frac{13}{2} \leq |F(S_n)| \leq 7$ implying that both the lower bound and upper bound are tight.

3 Concluding Remarks

The feedback vertex set problem is oriented from the circuit design. Recently, the related research focused on interconnection networks including meshes, toris, butterflies, and hypercubes are widely studied. Then, new bounds are established one after the other. The star graph is an attractive topologies having many nice properties than the mentioned graphs. In this paper, we set up the upper bound to the size of the feedback vertex set in star graphs by a constructive proof. We also give a formula, a modification of Focardi et al.[7], to show the lower bound of the feedback vertex set in k-regular graphs. Certainly, this bound suits star graphs and is shown to be sharp. However, the feedback vertex set we have found is not the minimum. An interested problem is to explore more precise bounds.

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