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THE CONSTRUCTION OF OPTIMAL K-FAULT-TOLERANT DESIGNS FOR TOKEN RINGS *

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ABSTRACT

Designing an optimal k-fault-tolerant network for token rings is equivalent to constructing an optimal k-hamiltonian graph, where k is a positive integer and corresponds to the number of faults. A graph G is k-hamiltonian if $G - F$ is hamiltonian for any set $F \subset (V \cup E)$ with $|F| \leq k$. An n-node k-hamiltonian graph is optimal if it contains the least number of edges among all n-node k-hamiltonian graphs. In this paper, we propose a construction scheme for the optimal k-hamiltonian graphs for every positive integer k. Applying this scheme, we can easily construct a family of optimal k-hamiltonian graphs with diameter $2\log_{d-1} n - O(1)$, where n is the number of vertices and d is the maximum degree. This diameter equals 2 times of Moore bound.

1 INTRODUCTION AND DEFINITIONS

Fault tolerance is essential in massively parallel systems that have a relatively high failure probability. A number of fault-tolerant designs for specific multiprocessor architectures have been proposed based on graph theoretic models in which the processor-to-processor interconnection structure is represented by a graph.

Let the graph $G = (V, E)$ represent an underlying interconnection network. Two types of failures in a multiprocessor system are of interest, processor failures and link failures. A link failure corresponds to the deletion of an edge from G, while a processor failure corresponds to the deletion of a node and all edges incident on it from G. If F is a set of faulty components including faulty nodes and faulty edges in G, then $G - F$ denotes the graph obtained by deleting the fault set F from G. To be specific, $F = V_1 \cup E_1$ for $E_1 \subset E$ and $V_1 \subset V$. We use $G - F$ to denote the graph $G' = (V - V_1, (E - E_1) \cap ((V - V_1) \times (V - V_1)))$. Note that a link fault cannot be ascribed to a fault at one of the adjacent processors since the adjacent processors of a faulty link are still included in reconfigurations while faulty processors are not. Most of previous research in designing optimal fault-tolerant topologies were concentrated on the cases that only processor failures were allowed [1, 2, 5, 8], or only link failures were allowed [3, 7, 9, 10, 14, 15, 17, 19, 20]. In their constructions, a supergraph $G' = (V', E')$ with respect to G is constructed such that $G' - F$ contains G as a subgraph where F is a set of faulty components with restriction to either $F \subset V'$ or $F \subset E'$. On the other hand, we consider $F \subset V \cup E$ that is any combination of processor failures and link failures of $G = (V, E)$. Our design concern is that $G - F$ contains a specified network topology that includes all nonfaulty processors. Henceforth, by "k faults" we mean k-component faults in any combination of processor failures and link failures. In this paper, we aim at designing *k-fault-tolerant* networks G for *token rings*, that is, for any k-fault set F, $G - F$ con-

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tains a token ring including all of the nonfaulty processors. Furthermore, our constructions are shown to be optimal in terms of the number of edges contained in G. Note that token rings contained in $G-F$ may contain different number of processors for different k-fault F.

A *path* is a sequence of nodes such that two consecutive nodes are adjacent. A path is delimited by "$\langle$" and "$\rangle$", for example, $\langle v_0, v_1, \cdots, v_{t-1}\rangle$. The path $\langle v_0, v_1, v_2, \ldots, v_{t-1}\rangle$ is also written as $\langle v_0 \rightarrow P_1 \rightarrow v_i, v_{i+1}, \ldots, v_j \rightarrow P_2 \rightarrow v_k, v_{k+1}, \cdots, v_{t-1}\rangle$, where $P_1 = \langle v_0, v_1, \ldots, v_i\rangle$, $P_2 = \langle v_j, v_{j+1}, \cdots, v_k\rangle$, and $j \geq i$. A cycle is a path, with at least three nodes, whose first node and last node are the same. A *hamiltonian cycle* is a cycle whose nodes are distinct and span all nodes. A *hamiltonian graph* is a graph that has a hamiltonian cycle. Let k be a positive integer. A graph G is *k-hamiltonian* if $G-F$ is hamiltonian for any set $F \subset V \cup E$ with $|F| \leq k$. It follows that a k-hamiltonian graph is a k-fault-tolerant network for token rings since $G-F$ contains a token ring that includes all of the nonfaulty processors for any k-fault set F. The design of *k-fault-tolerant* networks for token rings is equivalent to the design of k-hamiltonian graphs. It is obvious that a k-hamiltonian graph has at least $k+3$ nodes. Moreover, the degree of any node in a k-hamiltonian graph is at least $k+2$. An n-node k-hamiltonian graph is *optimal* if it contains the least number of edges among all n-node k-hamiltonian graphs. Henceforth, we use n to denote the number of vertices in a graph.

In [7, 8], Harary and Hayes presented a family of optimal 1-hamiltonian graphs whose diameter is $\lfloor \frac{n+1}{4} \rfloor$. Mukhopadhyaya and Sinha [12] proposed a family of optimal 1-hamiltonian graphs. The diameter of any graph in this family is $\lfloor \frac{n}{6} \rfloor + 2$ if n is even and $\lfloor \frac{n}{8} \rfloor + 3$ if n is odd. Wang et al. [18] proposed another family of optimal 1-hamiltonian graphs with diameter $O(\sqrt{n})$. It is natural to ask whether we can find optimal 1-hamiltonian graphs with a smaller diameter. This problem relates to the famous (n, d, D) problem in which we want to construct a graph of n nodes with maximum degree d such that the diameter D is minimized. When d and n are given, the lower bound on diameter D, called the *Moore bound*, is given by $D \geq \log_{d-1} n - \frac{2}{d}$ [4].

An undirected graph G is called *circulant graph* with distance sequence $\{d_1, d_2, \cdots, d_k\}$ if $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(G) = \{(v_i, v_j) \mid (i-j) \bmod n = d_l, \forall 1 \leq l \leq k\}$. Given two positive integers n and k with $n > 2k$, we construct a graph $G_{n,k}$ as follows: The nodes of $G_{n,k}$ are denoted by $x_0, x_1, \ldots, x_{n-1}$ and are arranged clockwise with the ascending order of the indices. If k is even, $G_{n,k}$ is defined as a circulant graph with distance sequence $\{1, 2, \ldots, \frac{k}{2}+1\}$. If k is odd and n is even, $G_{n,k}$ is defined as a circulant graph with distance sequence $\{1, 2, \ldots, \frac{k+1}{2}, \frac{n}{2}\}$. Otherwise, $G_{n,k}$ is not a circulant graph but has the edge set $\{(x_i, x_{i+j}) \mid 0 \leq i \leq n-1 \text{ and } 1 \leq j \leq \frac{k+1}{2}\} \cup \{(x_i, x_{i+\frac{n+1}{2}}) \mid 0 \leq i \leq \frac{n-3}{2}\} \cup \{(x_0, x_{\frac{n-1}{2}})\}$. Wang et al. [19] and Paoli et al. [14] proved that any $G_{n,k}$ is optimal k-node-hamiltonian and optimal k-edge-hamiltonian. Sung et al. [16] proved that any $G_{n,k}$ is optimal k-hamiltonian for $k=2$ and 3. They also conjectured that any $G_{n,k}$ is optimal k-hamiltonian. We note that the diameter of the family of $G_{n,k}$ is $O(n)$ if k is considered as a constant.

In this paper, we present a general scheme to construct optimal k-hamiltonian graphs. Applying this scheme, we can easily construct a family of optimal k-hamiltonian graphs with diameter $2\log_{d-1} n - O(1)$, that is 2 times of Moore bound.

2 HAMILTONICITY OF COMPLETE GRAPHS

In what follows, p is an integer with $p \geq 3$ and $k = p-2$. The degree of a vertex v in G is denoted by $deg_G(v)$. We use K_p to denote the complete graph(also called clique) that has p nodes.

Lemma 1 *Let F be any faulty edge set in K_p with $|F| \leq p-3 = k-1$. Then the graph $G = K_p - F$ has a hamiltonian cycle.*

Proof. Let x and y be two different nodes of K_p. The Ore's Theorem [13] state that a graph H of order p satisfies $deg_H(x) + deg_H(y) \geq p$ for every pair of non-adjacent vertices x, y in H; thus H has a hamiltonian cycle. It follows that $\deg_G(x) + \deg_G(y) \geq 2(p-2) - (p-4) \geq p$ for every pair of non-adjacent vertices x, y in G. Therefore, G has

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a hamiltonian cycle. Hence this lemma is proved. □

Corollary 1 *Let F be any faulty edge set in K_p with $|F| \leq k$. Then the graph $K_p - F$ has a hamiltonian path.*

Let $r \geq 2$ be an integer. A graph $G = (V, E)$ is r-*HP* if there exists $V' \subseteq V$ with $|V'| = r$ such that every pair of nodes in V' can be joined by a hamiltonian path of G.

Theorem 1 *$K_p - F$ is $(p-f)$-HP for any $F \subset V \cup E$ with $|F| = f \leq k$.*

Proof. We prove this theorem by induction on p. This theorem is obviously true for $p = 3$ and 4. Assume that it is true for all K_t for $3 \leq t < p$ and $p \geq 5$.

First, we consider that $|F \cap V| = i > 0$. Then the graph $K_p - F$ is isomorphic to $K_{p-i} - F^*$ for some $|F^*| \leq f - i$. By induction hypotheses, $K_{p-i} - F^*$ is $(p-f)$-HP. Hence $K_p - F$ is $(p-f)$-HP.

Next, we consider that $F \subset E$. When $|F| = k = p-2$, it follows from Corollary 1 that $K_p - F$ has a hamiltonian path, and thus $K_p - F$ is 2-HP.

Now consider that $F \subset E$ and $|F| \leq p-3$. Let H denote the subgraph given by (V, F) of K_p generated by F. We have $\sum_{v \in V} \deg_H(v) \leq 2(p-3)$. Thus, there exists a vertex $v \in V$ with $deg_H(v) \leq 1$.

Case 1: there exists a node v with $\deg_H(v) = 0$.

In other words, all the edges incident with v are fault-free. Thus, F is in $K_p - v$ and $K_p - v - F$ is isomorphic to $K_{p-1} - F$. By induction hypotheses, $K_p - v - F$ is $(p-1-f)$-HP. Therefore, there exists a subset $Y \subseteq V - \{v\}$ with $|Y| = p - 1 - f$ such that every two distinct nodes $x, y \in Y$ can be joined by a hamiltonian path of $K_p - v - F$. Let P be a hamiltonian path of $K_p - v - F$ joining x and y which can be written as $\langle x, x' \to P' \to y\rangle$ where x' is a node adjacent to x and P' is a path from x to y. Then $\langle x, v, x' \to P' \to y\rangle$ and $\langle v, x, x' \to P' \to y\rangle$ form two hamiltonian paths of $K_p - F$ joining x, y and v, y, respectively. Since x and y are arbitrary nodes in Y, we can conclude that there exists a hamiltonian path of $K_p - F$ joining any two different nodes in $Y \cup \{v\}$. Thus, $K_p - F$ is $(p-f)$-HP.

Case 2: there exists a node v with $\deg_H(v) = 1$.

In other words, exactly one edge incident with v is faulty. Thus, $K_p - v - F$ is isomorphic to $K_{p-1} - F^*$ where $|F^*| = f - 1$. By induction hypotheses, $K_p - v - F$ is $(p-f)$-HP. Thus, there exists a subset $Y \subseteq V - \{v\}$ with $|Y| = p - f$ such that every two distinct nodes $x, y \in Y$ can be joined by a hamiltonian path of $K_p - v - F$. Let P be a hamiltonian path of $K_p - v - F$ joining x and y, which can be written as $\langle x = z_0, z_1, \ldots, z_{p-2} = y\rangle$. Since $p \geq 5$, there exists z_j, $0 \leq j \leq p-3$, such that $(v, z_j) \notin F$ and $(v, z_{j+1}) \notin F$. Then $\langle x = z_0, z_1, \ldots, z_j, v, z_{j+1}, \ldots, z_{p-2} = y\rangle$ forms a hamiltonian path of $K_p - F$ joining x and y. In other words, $K_p - F$ is $(p-f)$-HP.

This theorem is proved. □

3 CONSTRUCTION SCHEME

Let x be a vertex of $G = (V, E)$ and $deg_G(x) = p$. All the vertices adjacent with x can be denoted by $x_1, x_2, \ldots, x_p$. Let $Z = (Y, W)$ be a p nodes clique whose vertex set $Y = \{y_1, y_2, \ldots, y_p\}$ and edge set $W = \{(y_1, y_j) | i \neq j\}$. The *$p$-node expansion* of G on x, which is denoted by $EXP_p(G, x)$, is a graph that is obtained from G by replacing x by the clique Z. To be specific, the graph $EXP_p(G, x) = (V^*, E^*)$ in which $V^* = (V - \{x\}) \cup Y$ and $E^* = E \cup W \cup \{(x_i, y_i) | 1 \leq i \leq p\} - \{(x, x_i) \mid 1 \leq i \leq p\}$. The graph $EXP_p(G, x)$ is obviously p-regular. K_5 and $EXP_4(K_5, x)$ are shown in Figure 1.

Theorem 2 *Let x be a vertex of $G = (V, E)$ and $deg_G(x) = p$. If G is k-hamiltonian, $EXP_p(G, x) = (V^*, E^*)$ is k-hamiltonian.*

Proof: Let F^* be a subset of $V^* \cup E^*$, where $|F^*| \leq k$. Let α be the cardinality of the set $F^* \cap (Y \cup W)$. Thus, $\alpha \leq k$. It follows from Theorem 1 that there exists a set $Y' \subseteq Y$ of size $(p - \alpha)$ such that every two distinct nodes in Y' can be joined by a hamiltonian path of the graph $Z - F^*$.

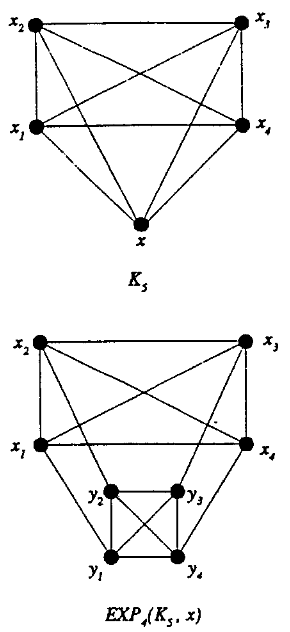

Figure 1: The graphs K_5 and $EXP_4(K_5, x)$.

We define a faulty set F of G as follows: $F = (F^* - (Y \cup W \cup \{(x_i, y_i) | 1 \le i \le p\})) \cup \{(x, x_i) \mid y_i \notin Y'\}$. Since $|F| \le |F^* - (Y \cup W)| + |\{(x, y_i) | y_i \notin Y'\}|$, it follows that $|F| \le |F^*| \le k$. Since G is k-hamiltonian, there exists a hamiltonian cycle $C = \langle x_i, x, x_j \to P \to x_i \rangle$ in the graph $G - F$ in which P is a path from x_j to x_i. By the definition of F, y_i and y_j are in Y'. Thus, there exists a hamiltonian path Q joining y_i and y_j of the graph $Z - (F^* \cap (Y \cup W))$. Therefore, $\langle x_i, y_i \to Q \to y_j, x_j \to P \to x_i \rangle$ forms a hamiltonian cycle in the graph $EXP_p(G, x) - F^*$. Thus, $EXP_p(G, x)$ is k-hamiltonian and this theorem is proved. □

Applying Theorem 2, we can easily obtain other k-hamiltonian graphs from a known k-hamiltonian graph by p-node expansion on a vertex of degree p. Note that the complete graph K_{p+1} is the smallest optimal k-hamiltonian graphs. For the same reason, graphs obtained from K_{p+1} by a sequence of p-node expansion are optimal k-hamiltonian. One possible sequence of p-node expansion is described by the following algorithm.

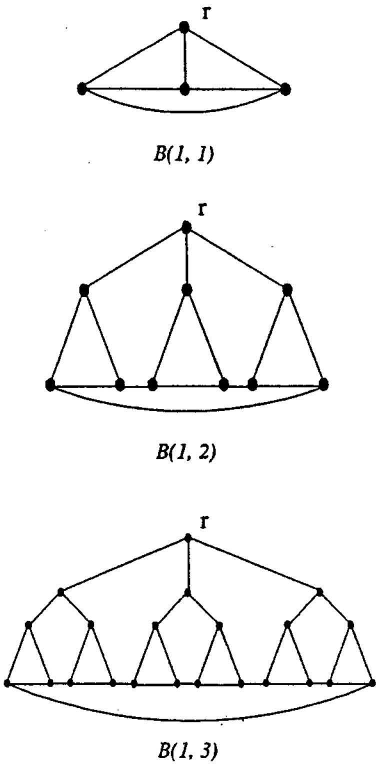

Figure 2: The graphs $B(1,1)$, $B(1,2)$, and $B(1,3)$.

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ALGORITHM Bell(p, s)
  G ← K_{p+1}
  pick any vertex r as the root of G
   for i ← 1 to s − 1 do
     B ← {v|distance(v, r) = i}
       for all v ∈ B
         G ← EXP_p(G, v)
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Let $B(k, s)$ denote the optimal k-hamiltonian graph obtained by Bell(p, s). The graphs $B(1,1), B(1,2)$, and $B(1,3)$ are shown in Figure 2. The node labeled with r indicates the root assigned by Bell(p, s). It can be verified that the number of nodes in $B(k, s)$ is $1 + \frac{p(p-1)^s - p}{p-2}$. Moreover, the distance between a node v to the root r is at most s. The diameter of $B(k, s)$ is at most $2s$. In other words, we have constructed a family of optimal k-hamiltonian graphs with diameter $2 \log_{p-1} n - O(1)$, that is 2 times of Moore bound.

4 CONCLUDING REMARKS

In this paper, we have presented a general scheme to construct optimal k-hamiltonian graphs. Furthermore, we use this scheme to construct $B(k,s)$ whose diameter is less than previous results in [7, 8, 18]. It would be interesting to find other families of optimal k-hamiltonian graphs whose diameter is smaller than that of $B(k,s)$. In Theorem 2, we prove that if G is k-hamiltonian, $EXP_p(G,x)$ is k-hamiltonian. In addition, we feel that the converse part of this theorem is also true:

Conjecture 1 *Let x be a vertex of $G=(V,E)$ and $deg_G(x)=p$. If $EXP_p(G,x)$ is k-hamiltonian then G is k-hamiltonian, for $k=p-2$.*

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