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# The Birnbaum Importance and the Optimal Replacement for the Consecutive-k-out-of-n:F System

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## ABSTRACT

This paper deals with the problem that given a new component and a consecutive-k-out-of-n:F system, which component should be replaced with the new one such that the resulting system reliability is maximized. Based on the B-importance, we propose an O(n) time algorithm for the linear system and an $O(n^2)$ time algorithm for the circular system. Both algorithms are far more efficient than the straightforward algorithms. As a special case, if the reliability of the original circular system is 0, the optimal replacement for this system can be found in O(kn) time by our modified circular algorithm.

## 1. INTRODUCTION

The reliability of the consecutive-k-out-of-n:F system was studied by many people [1-8]. So far, the time complexities of the best algorithms for computing the reliabilities of the linear system and the circular system are O(n) [4] and O(kn) [6,7,8], respectively.

Consider the problem: "Given a new component and a consecutive-k-out-of-n:F system, which component should be replaced with the new one such that the resulting system reliability is maximized?" When a component is replaced, the change of system reliability is not only dependent on the reliabilities of the removed component and the new component, but also on the reliabilities of all other components. A straightforward algorithm is re-computing the reliabilities of every possible replacement using algorithms in [4,6,7,8] and selecting the best. The time complexities of the straightforward algorithms for the linear system and the circular system are $O(n^2)$ and $O(kn^2)$, respectively. In this paper, we propose an O(n) algorithm for the linear system and an $O(n^2)$ algorithm for the circular system. As a special case, if the reliability of the original circular system is 0, the optimal replacement for this system can be found in O(kn) time by our modified circular algorithm.

### 1.1 Assumptions

<1> All components are mutually s-independent.

<2> Component 1,2,...,n are arranged in a line (or a circle) in that order for a linear (or circular) system.

### 1.2 Notation

| | |
|---|---|
| $p_i$ | the reliability of component i. |
| $q_i$ | $1-p_i$. |
| $p^*$ | the reliability of the new component. |
| $R_L(i,j)$ | the reliability of the linear system which consists of component i,i+1,...,j. Indices are taken modulo n. |
| $R_C(i,j)$ | the reliability of the circular system which consists of component i,i+1,...,j. Indices are taken modulo n. |
| $R^x{}_L(i,j)$ | the reliability of the linear system after replacing component x by the new component. |
| $R^x{}_C(i,j)$ | the reliability of the circular system after replacing component x by the new component. |
| $E_i$ | the event that component i functions. |
| $\bar{E}_i$ | the event that $E_i$ doesn't occur. |
| $S_L$ | the event that the linear system which consists of component 1,2,...,n functions. |
| $S_C$ | the event that the circular system which consists of component 1,2,...,n functions. |

$I_L(i)$ the B-importance of component i in the linear system.

$I_C(i)$ the B-importance of component i in the circular system.

$\Delta R^i{}_L$ $I_L(i)(p^*-p_i)$

$\Delta R^i{}_C$ $I_C(i)(p^*-p_i)$

## 2. OPTIMAL REPLACEMENT

The B-importance [9] is a useful tool in solving the optimal replacement problem. The B-importance of component i is defined as

$$I_L(i) = \frac{\partial R_L(1,n)}{\partial p_i}$$
$$I_C(i) = \frac{\partial R_C(1,n)}{\partial p_i}$$

And Papastavridis [9] proved that

$$I_L(i) = \frac{R_L(1,i-1)R_L(i+1,n) - R_L(1,n)}{q_i} \quad (1)$$
$$I_C(i) = \frac{R_L(i+1,i-1) - R_C(1,n)}{q_i} \quad (2)$$

The following lemma illustrates that $I_L(i), I_C(i)$ are independent of $p_i$.

**Lemma 1**

$I_L(i), I_C(i)$ are independent of $p_i$.

**Proof:**

In Papastavridis's proof of equation (3), (4) in [9],

$$I_L(i) = \Pr\{S_L \mid E_i\} - \Pr\{S_L \mid \bar{E}_i\}$$
$$I_C(i) = \Pr\{S_C \mid E_i\} - \Pr\{S_C \mid \bar{E}_i\}$$

Obviously, the right hand sides of the above two equations are independent of $p_i$.

□

Here we propose two new equations, which express both linear and circular reliabilities of a consecutive-k-out-of-n:F system in terms of B-importances. And with these equations, the optimal replacement is derived in Theorem 3.

**Lemma 2**

$$R_L(1,n) = I_L(i)p_i + A_i$$
$$R_C(1,n) = I_C(i)p_i + B_i$$

, where both $A_i$ and $B_i$ are independent of $p_i$.

**Proof:**

$$\begin{aligned}
&R_L(1,n)\\
&= \Pr\{S_L\}\\
&= \Pr\{S_L \cap E_i\} + \Pr\{S_L \cap \bar{E}_i\}\\
&= \Pr\{S_L \mid E_i\}\Pr\{E_i\} + \Pr\{S_L \mid \bar{E}_i\}\Pr\{\bar{E}_i\}\\
&= \Pr\{S_L \mid E_i\}p_i + \Pr\{S_L \mid \bar{E}_i\}(1-p_i)\\
&= [\Pr\{S_L \mid E_i\} - \Pr\{S_L \mid \bar{E}_i\}]p_i + [\Pr\{S_L \mid \bar{E}_i\}]\\
&= I_L(i)p_i + A_i, \quad A_i = \Pr\{S_L \mid \bar{E}_i\}.
\end{aligned}$$

The linear case is hereby proved, and the circular case can be proved similarly.

□

**Theorem 3**

The optimal replacement is to replace component i with the new one such that $\Delta R^i{}_L$ ($\Delta R^i{}_C$), for a linear (circular) system, is maximized.

**Proof:**

Consider the linear case first.

$$\begin{aligned}
&R^i{}_L(1,n) - R_L(1,n)\\
&= (I_L(i)p^* + A_i) - (I_L(i)p_i + A_i)\\
&= I_L(i)(p^* - p_i)\\
&= \Delta R^i{}_L
\end{aligned}$$

Clearly the optimal replacement is to replace component i such that $\Delta R^i{}_L$ is maximized. The linear case is hereby proved, and a similar proof can be done for the circular case.

□

We then propose two algorithms based on Theorem 3 for the optimal replacement problems for the linear and circular systems, and the time complexities of these algorithms are also given below.

**Linear algorithm**

<1> Compute $R_L(1,1), R_L(1,2),..., R_L(1,n)$.

<2> Compute $R_L(2,n), R_L(3,n),..., R_L(n,n)$.

<3> Compute $I_L(i)$ for i=1,2,…,n. ( by equation (1) ).

<4> Compute $\Delta R^i{}_L = I_L(i)(p^* - p_i)$, for i=1,2,…,n.

<5> Output the optimal i and $R_L(1,n)+\Delta R^i{}_L$ such that $\Delta R^i{}_L$ is maximized.

**Time complexity for linear algorithm**

Step 1 takes O(n) time, Step 2 O(n) time, Step 3 O(n) time, Step 4 O(n) time, and Step 5 O(1) time. The total time complexity is O(n).

□

**Circular algorithm**

<1> Compute $R_C(1,n)$.

<2> Compute $R_L(2,n), R_L(3,1),..., R_L(n,n-2), R_L(1,n-1)$.

<3> Compute $I_C(i)$ for i=1,2,…,n. ( by equation (2) ).

<4> Compute $\Delta R^i{}_C = I_C(i)(p^* - p_i)$, for i=1,2,…,n.

<5> Output the optimal i and $R_C(1,n)+\Delta R^i{}_C$ such that $\Delta R^i{}_C$ is maximized.

**Time complexity for circular algorithm**

Step 1 takes O(kn) time, Step 2 O($n^2$) time, Step 3 O(n) time, Step 4 O(n) time, and Step 5 O(1) time. The total time complexity is O($n^2$).

□

Since $I_L(i) \ge 0$, $I_C(i) \ge 0$ for all i, in both algorithms, we may only consider those i's with $p_i < p^*$ in step 4 and skip the i-related calculations in steps 2 and 3.

## 3. OPTIMAL REPLACEMENT FOR THE CIRCULAR 0-RELIABILITY SYSTEM

A 0-reliability consecutive-k-out-of-n:F system is a special consecutive-k-out-of-n:F system with reliability 0. This kind of system is practical if the system contains some failed components (with reliability 0) such that the system fails (with reliability 0). In this section we focus on the circular 0-reliability system.

**Theorem 4**

If $I_C(i) > 0$ for some i in a circular 0-reliability system C, then

<1> C has just one run of consecutive 0-reliability components with length at least k.

<2> This run contains fewer than 2k components.

<3> If this run contains component 1,2,…,m, ($k \le m < 2k$), then

$I_C(i) > 0$ for all $i \in \{m-k+1, m-k+2,..., k\}$.

$I_C(i) = 0$ for all $i \notin \{m-k+1, m-k+2,..., k\}$.

**Proof**

**Condition 1**: If C has no run of consecutive 0-reliability components with length at least k, C cannot be a 0-reliability system. If C has two or more run of consecutive 0-reliability components with length at least k, C is sure to be a 0-reliability system after replacing anyone of its components, that is, there is no i such that $I_C(i) > 0$.

**Condition 2**: If C has a run of consecutive 0-reliability components with length at least 2k, then C is sure to be a 0-reliability system after replacing anyone of its components, that is, there is no i such that $I_C(i) > 0$.

**Condition 3**: From condition 1 and 2, C must contain a run of consecutive 0-reliability components with length m, $(k \le m < 2k)$. If $i \in \{m-k+1, m-k+2, ..., k\}$, replacing component i in C with the new one will make C become a positive reliability system, that is, $I_C(i) > 0$. If $i \notin \{m-k+1, m-k+2, ..., k\}$, replacing component i in C with the new one does not change the system reliability, that is, $I_C(i) = 0$.

□

**Theorem 5**

In a circular 0-reliability system C, if $R^i c(1,n) > 0$, then
$R^i c(1,n) = R_L(i+1, i-1)p^*$.

**Proof**

$$\begin{aligned} & R^i c(1,n) \\ & = 0 + \Delta R^i c \\ & = 0 + I_C(i)(p^* - 0) \\ & = R_L(i+1, i-1)p^* \end{aligned}$$

□

**Modified circular algorithm (for the circular 0-reliability system)**

<1> Find the largest run of consecutive 0-reliability components and re-index components such that this run contains component 1,2,...,m, $m \ge k$.

<2> If $m \ge 2k$ then any replacement is optimal, stop.

<3> Compute $R_L(i+1, i-1)$ for $i \in \{m-k+1, m-k+2, ..., k\}$.

<4> Output the optimal i such that $R_L(i+1, i-1)$ is maximized, and output $R_L(i+1, i-1)p^*$, the optimal system reliability after replacing component i with the new one.

**Time complexity analysis for the Modified circular algorithm**

Step 1 takes O(n) time, Step 2 O(1) time, Step 3 O(kn) time, Step 4 O(1) time. The total time complexity is O(kn).

□

The modified circular algorithm and the original circular algorithm can be combined to solve the optimal replacement problem of a general circular system. When solving the problem, we use the original circular algorithm first. If $R_C(1,n) > 0$ in step 1, we can continue our computation in step 2-5. If $R_C(1,n) = 0$ in step 1, we can quit the computation of the original circular algorithm and use the modified circular algorithm, which is more efficient than the original.

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