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# The Clique-Transversal and the Clique-Independent Set Problems on Distance-Hereditary Graphs †

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***Abstract*** In this paper, we show that minimum clique-transversal and maximum clique-independent sets of a distance-hereditary graph have the same cardinality, and the clique-transversal set problem can be solved in $O(n+m)$ time and the clique-independent set problem can be solved in $O(n^2)$ time for distance-hereditary graphs.

**Keywords:** Algorithm, Clique-Transversal Set, Clique-Independent Set, Distance-Hereditary Graph.

## 1. Introduction

Let $G = (V, E)$ be a finite, simple, undirected graph with $|V| = n$ and $|E| = m$. A *clique* is a subset of pairwise adjacent vertices of $V$. A *maximal clique* is a clique that is not a proper subset of any other clique. A *clique-transversal set* of $G$ is a subset of vertices intersecting all maximal cliques of $G$. The *clique-transversal set problem* is to find a clique-transversal set of $G$ of minimum cardinality. The cardinality $\tau_C(G)$ of a minimum clique-transversal set of $G$ is called the *clique-transversal number* of $G$. A *clique-independent set* of $G$ is a collection of pairwise disjoint maximal cliques. The *clique-independent set problem* is to find a clique-independent set of $G$ of maximum cardinality. The cardinality $\alpha_C(G)$ of a maximum clique-independent set of $G$ is called the *clique-independence number* of $G$. It is clear that the *weak duality inequality* $\alpha_C(G) \leq \tau_C(G)$ holds for any graph $G$.

The clique-transversal set problem is a special case of the generalized clique-transversal problem [9], the clique $r$-domination problem [7], and $k$-fold clique-transversal problem [11], respectively. The clique-independent set problem is a special case of the clique $r$-packing problem [7]. Following the algorithm of [11] with $k = 1$, the clique-transversal set problem is polynomial-time solvable on balanced graphs. In [7], an efficient algorithm was proposed to solve the clique $r$-domination problem and the clique $r$-packing problem on dually chordal graphs. We can use this algorithm to solve the clique-transversal and the clique-independent set problems on a dually chordal graph, but the time complexity of this algorithm is proportional to the sum of the sizes of all maximal cliques of a dually chordal graph. Notice that a dually chordal graph may have a exponential number of maximal cliques.

The clique-transversal set problem has been widely studied in [1, 2, 3, 14, 19, 21]. Eades et al. [13] showed that the problem of deciding whether a chordal graph has two disjoint minimum clique-transversal sets is NP-complete. Both the clique-transversal and the clique-independent set problems are NP-hard for co-comparability graphs, planar graphs, line graphs, total graphs, split graphs, undirected path graphs, and $k$-trees with unbounded $k$ [8, 9, 15]. Furthermore, both problems are polynomial-time solvable for comparability graphs, strongly chordal graphs, and Helly circular-arc graphs [4, 8, 9, 15]. In [20], Sheu extended the algorithm of [4] to solve the weighted version of the clique-transversal set problem on weighted comparability graphs in $O(m\sqrt{n} + M(n))$ time, where $M(n)$ is the complexity of multiplying two $n \times n$ matrices.

A graph $G$ is *clique-perfect* if $\tau_C(F) = \alpha_C(F)$ for every induced subgraph $F$ of $G$ [15]. The following are examples of clique-perfect graph classes: chordal graphs without odd suns [18], strongly chordal graphs [8], and

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comparability graphs [4]. Durán [12] et al. demonstrated that $\tau_C(G)$ and $\alpha_C(G)$ can be computed in polynomial time for any clique-perfect graph $G$ by using integer linear programming. A graph $G = (V, E)$ is called *distance-hereditary* if every pair of vertices are equidistant in every connected induced subgraph containing them. For other features of distance-hereditary graphs, please refer to [5, 6, 16]. It has been shown [17] that finding a *minimum-weighted* clique-transversal set on weighted distance-hereditary graphs can be solved in $O(n + m)$ time and the clique-independence number of a distance-hereditary graph can be computed in $O(n^3)$ time, but it remains open whether distance-hereditary graphs are clique-perfect. In this paper, we show that $\tau_C(G) = \alpha_C(G)$ for any distance-hereditary graph $G$, and the clique-transversal set problem can be solved in $O(n+m)$ time and the clique-independent set problem can be solved in $O(n^2)$ time for distance-hereditary graphs. Following the definition of distance-hereditary graphs, every induced subgraph of a distance-hereditary graph is *distance-hereditary*, too. The equation $\tau_C(F) = \alpha_C(F)$ holds for every induced subgraph $F$ of $G$. Therefore, distance-hereditary graphs are clique-perfect.

## 2. Preliminaries

The following theorem shows that distance-hereditary graphs can be defined recursively.

**Theorem 1.** [10] Distance-hereditary graphs can be defined recursively as follows:

1. A graph consisting of only one vertex is distance-hereditary, and the twin set is the vertex itself.
2. If $G_1$ and $G_2$ are disjoint distance-hereditary graphs with the twin sets $TS(G_1)$ and $TS(G_2)$, respectively, then the graph $G = G_1 \cup G_2$ is a distance-hereditary graph and the twin set of $G$ is $TS(G_1) \cup TS(G_2)$. $G$ is said to be obtained from $G_1$ and $G_2$ by a *false twin* operation.
3. If $G_1$ and $G_2$ are disjoint distance-hereditary graphs with the twin sets $TS(G_1)$ and $TS(G_2)$, respectively, then the graph $G$ obtained by connecting every vertex of $TS(G_1)$ to all vertices of $TS(G_2)$ is a distance-hereditary graph, and the twin set of $G$ is $TS(G_1) \cup TS(G_2)$. $G$ is said to be obtained from $G_1$ and $G_2$ by a *true twin* operation.
4. If $G_1$ and $G_2$ are disjoint distance-hereditary graphs with the twin sets $TS(G_1)$ and $TS(G_2)$, respectively, then the graph $G$ obtained by connecting every vertex of $TS(G_1)$ to all vertices of $TS(G_2)$ is a distance-hereditary graph, and the twin set of $G$ is $TS(G_1)$. $G$ is said to be obtained from $G_1$ and $G_2$ by a *pendant vertex* operation. (In the rest of the paper, we assume that $TS(G) = TS(G_1)$ whenever we say that $G$ is obtained from $G_1$ and $G_2$ by a pendant vertex operation.)

By Theorem 1, a distance-hereditary graph $G$ has its own twin set $TS(G)$, the twin set $TS(G)$ is a subset of vertices of $G$, and it is defined recursively. The construction of $G$ from disjoint distance-hereditary graphs $G_1$ and $G_2$ as described in Theorem 1 involves only the twin sets of $G_1$ and $G_2$.

Following Theorem 1, a binary ordered decomposition tree can be obtained in linear-time [10]. In this decomposition tree, each leaf is a single vertex graph, and each internal node represents one of the three operations: pendant vertex operation (labelled by P), true twin operation (labelled by T), and false twin operation (labelled by F). This ordered decomposition tree is called a **PTF**-tree. It has $2n - 1$ tree nodes. Figure 1 illustrates an example of a **PTF**-tree. Hence, a **PTF**-tree of a distance-hereditary graph can be obtained in linear-time [10].

## 3. Distance-Hereditary Graphs Are Clique-Perfect

In this section, we will prove that distance-hereditary graphs are clique-perfect by induction. We observe that a graph $G$ of a single vertex holds the *duality equality* $\alpha_C(G) = \tau_C(G)$. Suppose that $G_1$ and $G_2$ are two distance-hereditary graphs that hold the duality equality. We will show that a graph $G$ obtained from $G_1$ and $G_2$ by any one of operations mentioned in Theorem 1 always holds the duality equality. It will reveal

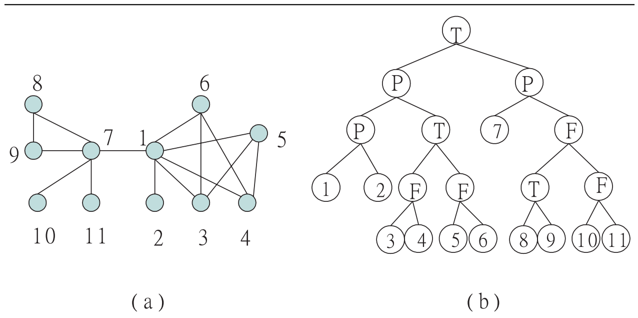


**Figure 1. (a) A distance-hereditary graph $G$. (b) A PTF-tree of $G$.**

that distance-hereditary graphs hold the duality equality. Since an induced subgraph of a distance-hereditary graph is also distance-hereditary, it will follow that distance-hereditary graphs are clique-perfect.

Throughout this section, we assume that $G = (V, E)$ is a distance-hereditary graph. For a subset $V'$ of $V$, $G[V']$ is the subgraph induced by $V'$. Before proving that distance-hereditary graphs are clique-perfect, we give some observations about maximal cliques based upon the recursive definition of distance-hereditary graphs.

**Definition 1.** We use $C(G)$ to denote the collection of all maximal cliques of $G$. Hence $C(G[TS(G)])$ is the collection of all maximal cliques of $G[TS(G)]$. We use $C_{TS}(G)$ to denote the collection of all maximal cliques of $G$ which are maximal cliques of $G[TS(G)]$ and use $C_{\overline{TS}}(G)$ to denote the collection of all maximal cliques of $G$ which are not maximal cliques of $G[TS(G)]$. Hence $C(G) = C_{TS}(G) \cup C_{\overline{TS}}(G)$. Let $C_E(G) = C(G) \cup C(G[TS(G)])$. $C_E(G)$ denotes the collection of all maximal cliques of $G$ and all maximal cliques of $G[TS(G)]$.

**Remark 1.** *Suppose that $G$ is a graph of single vertex and $v$ is the vertex of $G$. Then, $C(G) = C(G[TS(G)]) = C_{TS}(G) = C_E(G) = \{\{v\}\}$ and $C_{\overline{TS}}(G) = \emptyset$.*

**Remark 2.** *A maximal clique of $G[TS(G)]$ is not necessarily a maximal clique of $G$. If all maximal cliques of $G[TS(G)]$ are maximal cliques of $G$, then $C(G) = C_E(G)$. On the other hand, if all maximal cliques of $G[TS(G)]$ are not maximal cliques of $G$, then $C(G) = C_{\overline{TS}}(G)$.*

**Lemma 1.** *Suppose that $G$ is a graph obtained from two disjoint distance-hereditary graphs $G_1$ and $G_2$ by a false twin operation. Then, we have*

(1) $C(G[TS(G)]) = C(G_1[TS(G_1)]) \cup C(G_2[TS(G_2)])$,

(2) $C(G) = C(G_1) \cup C(G_2)$,

(3) $C_{TS}(G) = C_{TS}(G_1) \cup C_{TS}(G_2)$,

(4) $C_{\overline{TS}}(G) = C_{\overline{TS}}(G_1) \cup C_{\overline{TS}}(G_2)$, *and*

(5) $C_E(G) = C_E(G_1) \cup C_E(G_2)$.

***Proof.*** By definition. □

**Definition 2.** Suppose that $G$ is a graph obtained from two disjoint distance-hereditary graphs $G_1$ and $G_2$ by a true twin operation or a pendant vertex operation. We use $C_{12}(G)$ to denote $\{c_1 \cup c_2 | c_1 \in C(G_1[TS(G_1)])$ and $c_2 \in C(G_2[TS(G_2)])\}$.

**Lemma 2.** *Suppose that $G$ is a graph obtained from two disjoint distance-hereditary graphs $G_1$ and $G_2$ by a true twin operation. Then, we have*

(1) $C(G[TS(G)]) = C_{12}(G)$,

(2) $C(G) = C_{\overline{TS}}(G_1) \cup C_{\overline{TS}}(G_2) \cup C_{12}(G)$,

(3) $C_{TS}(G) = C(G[TS(G)])$,

(4) $C_{\overline{TS}}(G) = C_{\overline{TS}}(G_1) \cup C_{\overline{TS}}(G_2)$, *and*

(5) $C_E(G) = C(G)$.

***Proof.*** In the following, we just show the correctness of statement (1). The other statements of this lemma can be easily verified by definition. Notice that $G$ is obtained by connecting every vertex of $TS(G_1)$ to all vertices of $TS(G_2)$, and $TS(G) = TS(G_1) \cup TS(G_2)$. Clearly, every clique in $C_{12}(G)$ is also a clique of $G[TS(G)]$. Let $c$ be a maximal clique in $C(G[TS(G)])$, $c_1 = c \cap TS(G_1)$, and $c_2 = c \cap TS(G_2)$. Suppose that $c_1$ is not a maximal clique of $G_1[TS(G_1)]$. There exists a maximal clique $c_1'$ in $C(G_1[TS(G_1)])$ such that $c_1 \subset c_1'$. Then $c_1' \cup c_2$ is a clique of $G[TS(G)]$ and $c \subset (c_1' \cup c_2)$, which contradicts that $c$ is a maximal clique of $G[TS(G)]$. Therefore, $c_1 \in C(G_1[TS(G_1)]$. Similarly, we can prove that $c_2 \in C(G_2[TS(G_2)])$. Hence, $c \in C_{12}(G)$. Conversely, let $c_1$ be a maximal clique in $C(G_1[TS(G_1)])$ and $c_2$ be a maximal clique in $C(G_2[TS(G_2)])$. Then $c = c_1 \cup c_2$ is a clique in $C_{12}(G)$. Suppose that $c$ is not a maximal clique of $G[TS(G)]$. There exists a maximal clique $c' \in C(G[TS(G)])$ such that $c \subset c'$. Then either $c_1 \subset (c' \cap TS(G_1))$ or $c_2 \subset (c' \cap TS(G_2))$. However, either of them contradicts that $c_1$ and $c_2$ are maximal cliques of $G[TS(G_1)]$ and $G[TS(G_2)]$, respectively. Therefore, $(c_1 \cup c_2) \in C(G[TS(G)])$. Following the discussion above, $C(G[TS(G)]) = C_{12}(G)$. □

**Lemma 3.** *Suppose that $G$ is a graph obtained from two disjoint distance-hereditary graphs $G_1$ and $G_2$ by a pendant vertex operation. Then, we have*

(1) $C(G[TS(G)]) = C(G_1[TS(G_1)])$,

(2) $C(G) = C_{\overline{TS}}(G_1) \cup C_{\overline{TS}}(G_2) \cup C_{12}(G)$,

(3) $C_{TS}(G) = \emptyset$,

(4) $C_{\overline{TS}}(G) = C(G)$,

(5) $C_E(G) = C(G) \cup C(G_1[TS(G_1)])$.

***Proof.*** Notice that $G$ is obtained by connecting every vertex of $TS(G_1)$ to all vertices of $TS(G_2)$, but $TS(G) = TS(G_1)$. By arguments similar to those for proving Lemma 2, this lemma can be easily proved. □

**Lemma 4.** *Suppose that $G$ is a graph obtained from two disjoint distance-hereditary graphs $G_1$ and $G_2$ by a true twin operation or a pendant vertex operation. If $S$ is a clique-transversal set of $G$, then either $S \cap TS(G_1)$ is a clique-transversal set of $G_1[TS(G_1)]$ or $S \cap TS(G_2)$ is a clique-transversal set of $G_2[TS(G_2)]$.*

***Proof.*** Assume for contrary that neither $S \cap TS(G_1)$ is a clique-transversal set of $G_1[TS(G_1)]$ nor $S \cap TS(G_2)$ is a clique-transversal set of $G_2[TS(G_2)]$. There exist maximal cliques $c_1$ and $c_2$ of $G_1[TS(G_1)]$ and $G_2[TS(G_2)]$, respectively, such that $S$ does not contain any vertex in them. By Lemma 2 and Lemma 3, $c_1 \cup c_2$ is a maximal clique of $G$. However, $S$ does not contain any vertex in $c_1 \cup c_2$, which contradicts the assumption that $S$ is a clique-transversal set of $G$. □

To prove that distance-hereditary graphs are clique-perfect, we introduce the following definitions.

**Definition 3.** A *strong clique-transversal set* of $G$ is a subset of $V$ that intersects all cliques in $C_E(G)$. We use $SCT(G)$ to represent a strong clique-transversal set of $G$.

**Definition 4.** A *weak clique-transversal set* of $G$ is a subset of $V$ that intersects all maximal cliques in $C_{\overline{TS}}(G)$. We use $WCT(G)$ to represent a weak clique-transversal set of $G$.

**Definition 5.** A *weak clique-independent set* of $G$ is a collection of pairwise disjoint cliques in $C_{\overline{TS}}(G)$. We use $WCI(G)$ to represent a weak clique-independent set of $G$.

**Definition 6.** An *expanded clique-independent set* of $G$ is a collection of pairwise disjoint cliques in $C_E(G)$. We use $ECI(G)$ to represent an expanded clique-independent set of $G$.

**Definition 7.** Let $CT(G)$ and $CI(G)$ denote a clique-transversal set and a clique-independent set of $G$, respectively. We say that a distance-hereditary graph $G$ holds the *strong duality* if there exist a $CT(G)$, a $CI(G)$, a $WCT(G)$, a $WCI(G)$, an $SCT(G)$, and an $ECI(G)$ such that the following four conditions are satisfied:

(1) $|CT(G)| = |CI(G)|$,

(2) $|WCT(G)| = |WCI(G)|$,

(3) $|SCT(G)| = |ECI(G)|$, and

(4) $WCI(G) \subseteq ECI(G)$.

For simplicity, let $XI(G)$ denote $ECI(G) \setminus WCI(G)$.

**Remark 3.** *Suppose that $G$ holds the strong duality. Since $|WCT(G)| = |WCI(G)|$, such a $WCT(G)$ and a $WCI(G)$ are a minimum weak clique-transversal set and a maximum weak clique-independent set of $G$, respectively. Hence $XI(G) \subseteq C(G[TS(G)])$.*

**Remark 4.** *Since $|CI(G)| \leq \alpha_C(G) \leq \tau_C(G) \leq |CT(G)|$, $G$ holds the duality equality if there exist a clique-transversal set $CT(G)$ and a clique-independent set $CI(G)$ satisfying the condition that $|CT(G)| = |CI(G)|$.*

Instead of proving that distance-hereditary graphs hold the duality equality, we prove that they hold the strong duality. We will show how to find a $CT(G)$, a $CI(G)$, a $WCT(G)$, a $WCI(G)$, an $SCT(G)$, and an $ECI(G)$ such that the four conditions of strong duality are satisfied.

**Lemma 5.** *Assume that $G$ is a graph of single vertex and $v$ is the vertex of $G$. There exist the following sets:*

(1) $CT(G) = \{v\}$,

(2) $SCT(G) = \{v\}$,

(3) $WCT(G) = \emptyset$,

(4) $WCI(G) = \emptyset$,

(5) $ECI(G) = \{\{v\}\}$, *and*

(6) $CI(G) = \{\{v\}\}$

*such that $G$ holds the strong duality.*

***Proof.*** The lemma can be easily verified by the definition. □

**Definition 8.** Assume that $S$ is a family of sets. Let $\min S$ denote a set of minimum cardinality in $S$.

**Lemma 6.** *Assume that $G$ is formed from two disjoint distance-hereditary graphs $G_1$ and $G_2$ by a* **pendant vertex** *operation, and both $G_1$ and $G_2$ hold the strong duality. Suppose that $XI(G_1) = \{c_1, \ldots, c_{k_1}\}$, $XI(G_2) = \{d_1, \ldots, d_{k_2}\}$, and $k = \min\{k_1, k_2\}$. Let $X = \{c_i \cup d_i | 1 \leq i \leq k\}$. Let $\tilde{X} = \{c_{k+1}, \cdots, c_{k_1}\}$ if $k_1 > k$ and $\tilde{X} = \emptyset$ otherwise. There exist the following sets:*

(1) $CT(G) = \min\{SCT(G_1) \cup WCT(G_2), SCT(G_2) \cup WCT(G_1)\}$,

(2) $SCT(G) = SCT(G_1) \cup WCT(G_2)$,

(3) $WCT(G) = CT(G)$,

(4) $WCI(G) = WCI(G_1) \cup WCI(G_2) \cup X$,

(5) $ECI(G) = WCI(G_1) \cup WCI(G_2) \cup X \cup \tilde{X}$, *and*

(6) $CI(G) = WCI(G)$

*such that $G$ holds the strong duality.*

***Proof.***

(1) By (2) of Lemma 3, both $SCT(G_1) \cup WCT(G_2)$ and $SCT(G_2) \cup WCT(G_1)$ are clique-transversal sets of $G$. We let $CT(G) = \min\{SCT(G_1) \cup WCT(G_2), SCT(G_2) \cup WCT(G_1)\}$.

(2) Notice that $TS(G) = TS(G_1)$. Since $SCT(G_1)$ intersects all maximal cliques of $C(G[TS(G_1)])$, it intersects all maximal cliques of $C(G[TS(G)])$. Hence $SCT(G_1) \cup WCT(G_2)$ is a strong clique-transversal set of $G$. We let $SCT(G) = SCT(G_1) \cup WCT(G_2)$.

(3) By (4) of Lemma 3, a weak clique-transversal set of $G$ is also a clique-transversal set of $G$. We let $WCT(G) = CT(G)$.

Suppose that $XI(G_1) = \{c_1, \ldots, c_{k_1}\}$, $XI(G_2) = \{d_1, \ldots, d_{k_2}\}$, and $k = \min\{k_1, k_2\}$. Let $X = \{c_i \cup d_i | 1 \le i \le k\}$ and $\tilde{X} = \{c_{k+1}, \cdots, c_{k_1}\}$.

(4) By (2) and (4) of Lemma 3, $X$ is a clique-independent set of $G$ and $WCI(G_1) \cup WCI(G_2) \cup X$ is a weak clique-independent set of $G$. We let $WCI(G) = WCI(G_1) \cup WCI(G_2) \cup X$.

(5) It is easy to verify that $WCI(G_1) \cup WCI(G_2) \cup X \cup \tilde{X}$ is an expanded clique-independent set of $G$. We let $ECI(G) = WCI(G_1) \cup WCI(G_2) \cup X \cup \tilde{X}$.

(6) By (4) of Lemma 3, a clique-independent set of $G$ is also a weak clique-independent set of $G$. We let $CI(G) = WCI(G)$.

In the following, we show that $CT(G)$, $CI(G)$, $WCT(G)$, $WCI(G)$, $SCT(G)$, and $ECI(G)$ satisfy the conditions of strong duality.

Since $G_1$ and $G_2$ hold the strong duality, by (2) and (3) of Definition 7, $k_1 = |SCT(G_1)| - |WCT(G_1)| = |XI(G_1)|$ and $k_2 = |SCT(G_2)| - |WCT(G_2)| = |XI(G_2)|$. Clearly $|CT(G)| = |WCT(G)| = |WCT(G_1)| + |WCT(G_2)| + k = |WCI(G_1)| + |WCI(G_2)| + k = |WCI(G)| = |CI(G)|$. Next we verify that $|SCT(G)| = |ECI(G)|$. If $k = k_1$, then $\tilde{X}$ is empty, $|SCT(G)| = k_1 + |WCT(G_1)| + |WCT(G_2)|$, and $|ECI(G)| = |WCI(G_1)| + |WCI(G_2)| + k_1$. On the other hand, suppose that $k = k_2$. We have $|SCT(G)| = |SCT(G_1)| + |WCT(G_2)| = k_1 + |WCT(G_1)| + |WCT(G_2)|$ and $|ECI(G)| = |WCI(G_1)| + |WCI(G_2)| + |X| + |\tilde{X}| = |WCI(G_1)| + |WCI(G_2)| + k_1$. We can see that $|SCT(G)| = |ECI(G)|$ in both cases. Finally, $WCI(G) \subseteq ECI(G)$ is obvious. Thus $G$ holds the strong duality. □

**Lemma 7.** *Assume that $G$ is formed from two disjoint distance-hereditary graphs $G_1$ and $G_2$ by a* **true twin** *operation, and both $G_1$ and $G_2$ hold the strong duality. Suppose that $XI(G_1) = \{c_1, \ldots, c_{k_1}\}$, $XI(G_2) = \{d_1, \ldots, d_{k_2}\}$, and $k = \min\{k_1, k_2\}$. Let $X = \{c_i \cup d_i | 1 \le i \le k\}$. There exist the following sets:*

(1) $CT(G) = \min\{SCT(G_1) \cup WCT(G_2), SCT(G_2) \cup WCT(G_1)\}$,

(2) $SCT(G) = CT(G)$,

(3) $WCT(G) = WCT(G_1) \cup WCT(G_2)$,

(4) $WCI(G) = WCI(G_1) \cup WCI(G_2)$,

(5) $ECI(G) = WCI(G_1) \cup WCI(G_2) \cup X$, *and*

(6) $CI(G) = ECI(G)$

*such that $G$ holds the strong duality.*

***Proof***. By (2) and (5) of Lemma 2, we see that both $SCT(G_1) \cup WCT(G_2)$ and $SCT(G_2) \cup WCT(G_1)$ are not only clique-transversal sets of $G$ but also strong clique-transversal sets of $G$. Besides, $X$ is a clique-independent set of $G$ and $WCI(G_1) \cup WCI(G_2) \cup X$ is not only a clique-independent set of $G$ but also an expanded clique-independent set of $G$. Furthermore, by (4) of Lemma 2, $WCT(G_1) \cup WCT(G_2)$ and $WCI(G_1) \cup WCI(G_2)$ are a weak clique-transversal set and a weak clique-independent set of $G$, respectively. Therefore we let

(1) $CT(G) = \min\{SCT(G_1) \cup WCT(G_2), SCT(G_2) \cup WCT(G_1)\}$,

(2) $SCT(G) = CT(G)$,

(3) $WCT(G) = WCT(G_1) \cup WCT(G_2)$,

(4) $WCI(G) = WCI(G_1) \cup WCI(G_2)$,

(5) $ECI(G) = WCI(G_1) \cup WCI(G_2) \cup X$, and

(6) $CI(G) = ECI(G)$.

Since $G_1$ and $G_2$ hold the strong duality, by (2) and (3) of Definition 7, $k_1 = |SCT(G_1)| - |WCT(G_1)| = |XI(G_1)|$ and $k_2 = |SCT(G_2)| - |WCT(G_2)| = |XI(G_2)|$. Hence $|CT(G)| = |SCT(G)| = |ECI(G)| = |CI(G)| = k + |WCI(G_1)| + |WCI(G_2)|$. Besides, $|WCT(G)| = |WCI(G)|$. Finally, $WCI(G) \subseteq ECI(G)$ is obvious. Following the discussion above, $G$ holds the strong duality. □

**Lemma 8.** *Assume that $G$ is obtained from two disjoint distance-hereditary graphs $G_1$ and $G_2$ by a* **false twin** *operation, and both $G_1$ and $G_2$ hold the strong duality. There exist the following sets:*

(1) $CT(G) = CT(G_1) \cup CT(G_2)$,

(2) $SCT(G) = SCT(G_1) \cup SCT(G_2)$,

(3) $WCT(G) = WCT(G_1) \cup WCT(G_2)$,

(4) $WCI(G) = WCI(G_1) \cup WCI(G_2)$,

(5) $ECI(G) = ECI(G_1) \cup ECI(G_2)$, *and*

(6) $CI(G) = CI(G_1) \cup CI(G_2)$

*such that $G$ holds the strong duality.*

***Proof***. Following Lemma 1, $CT(G_1) \cup CT(G_2)$ is a clique-transversal set of $G$, and $SCT(G_1) \cup SCT(G_2)$ a strong clique-transversal set of $G$, …, etc. So we let

(1) $CT(G) = CT(G_1) \cup CT(G_2)$,

(2) $SCT(G) = SCT(G_1) \cup SCT(G_2)$,

(3) $WCT(G) = WCT(G_1) \cup WCT(G_2)$,

(4) $WCI(G) = WCI(G_1) \cup WCI(G_2)$,

(5) $ECI(G) = ECI(G_1) \cup ECI(G_2)$, and
(6) $CI(G) = CI(G_1) \cup CI(G_2)$.
Since $G_1$ and $G_2$ hold the strong duality, it is easy to verify that all four conditions of the strong duality are satisfied. □

Now we are ready to prove the main theorem.

**Theorem 2.** *Distance-hereditary graphs are clique-perfect.*

***Proof.*** We have explained the reasons that distance-hereditary graphs are clique-perfect if they hold the strong duality. Based upon the recursive definition of distance-hereditary graphs, and Lemmas 5, 6, 7, and 8, we can prove that distance-hereditary graphs hold strong duality by induction. Hence, distance-hereditary graphs are clique-perfect. □

## 4. Conclusion

We have shown distance-hereditary graphs are clique-perfect graphs. The proof is based upon the recursive definition of distance-hereditary graphs. Furthermore, as a byproduct of the inductive proof, we can design polynomial-time dynamic programming algorithms to find a minimum clique-transversal set and a maximum clique-independent set for a distance-hereditary graph $G$. The constructive proofs of Lemma 6, 7, and 8 suggest that we build a $CT(G)$ and a $CI(G)$ bottom up according to the PTF-tree of $G$. If we store a clique-transversal set in a linked list and a clique-independent set in a linked list of cliques, respectively. Then, the union of two clique-transversal sets or two clique-independent sets can be done in constant time. We see that the clique-transversal set problem can be solved in $O(n+m)$ and the clique-independent set problem can be solved in $O(n^2)$ time for distance-hereditary graphs. We conjecture that the algorithm for the clique-independent set problem on distance-hereditary graphs can be implemented in $O(n+m)$ time.

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