Stability Analysis of Neural-Network Large-scale Systems

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Abstract

This paper is concerned with the stability problem of a neural network (NN) large-scale system which consists of a few interconnected subsystems represented by NN models. First, the dynamics of each NN model is converted into LDI (linear differential inclusion) representation. Subsequently, based on the LDI representations, a stability criterion in terms of Lyapunov's direct method is derived to guarantee the asymptotic stability of NN large-scale systems. Finally, a numerical example with simulations is given to illustrate the results.

Key words— Large-scale system, neural network.

I. Introduction

A great number of today's problems are brought about by present-day technology and societal and environmental processes which are highly complex, large in dimension, and stochastic by nature. The field of large-scale systems exists so widely that covers either the fundamental theory of modeling, optimization, and control or certain particular aspects and applications. In addition, large-scale systems analysis, design, and control theory has attained considerable maturity and sophistication and is receiving increasing attention from theorists and practitioners, for both its methodological attraction and its important real-life applications [1]. In real systems, the large-scale systems include electric power systems, nuclear reactors, aerospace systems, large electric networks, economic systems, process control systems, chemical and petroleum industries, different types of societal systems, and ecological systems. Such systems consist of a number of interdependent subsystems which serve particular functions, share resources, and are governed by a set of interrelated goals and constraints [2]. Recently, many

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approaches have been used to investigate the stability and stabilization of large-scale systems, as proposed in the literature [3-6].

During the past several years, neural network (NN)-based modeling has become an active research field because of its unique merits in solving complex nonlinear system identification and control problems. Neural networks are composed of simple elements operating in parallel. These elements are inspired by biological nervous systems. Then, we can train a neural network to represent a particular function by adjusting the weights between elements. However, the stability analysis of nonlinear large-scale systems via NN model-based control is so difficult that rare researches were reported. Therefore, the LDI representation is employed in this study to deal with the stability analysis of nonlinear large-scale systems.

In this paper, we consider an NN large-scale system composed of a set of subsystems with interconnections. One critical property of control systems is stability and considerable reports have been issued in the literature on the stability problem of NN dynamic systems (see [7-9] and the references therein). However, as far as we know, the stability problem of NN large-scale systems remains unresolved. Hence, a stability criterion in terms of Lyapunov's direct method is derived to guarantee the asymptotic stability of NN large-scale systems. This paper may be viewed as a generalization of the approach of [8] to the stability analysis of NN large-scale systems.

This study is organized as follows. First, the system description is presented. Next, based on Lyapunov approach, a stability criterion is derived to guarantee the asymptotic stability of NN large-scale systems. Finally, a numerical example with simulations is given to demonstrate the results, followed by conclusions.

II. System description and stability analysis

Consider a neural-network (NN) large-scale system ***N*** which consists of J subsystems with interconnections. In addition, the jth isolated NN subsystem (without interconnection) of ***N***, shown in Fig. 1, has S_j ($j = 1, 2, \cdots, J$) layers with R^{τ} ($\tau = 1, 2, \cdots, S_j$) neurons for each layer, in which $x_j(k) \sim x_j(k-m+1)$ are the state variables. It is assumed that v is the net input and all the transfer functions $T(v)$ of units in the jth isolated NN subsystem are described by the following sigmoid function:

$$T(v)=\lambda\left(\frac{2}{1+\exp\left(-\dfrac{v}{\varsigma}\right)}-1\right), \tag{1}$$

where ς, $\lambda > 0$ are the parameters associated with the sigmoid function. In addition, we need to introduce some additional notations to specify these layers. The superscripts are used for identifying the layers. Specifically, we append the number of the layer as a superscript to the names of the variables. Thus, the weight matrix for the nth layer is written as W^n, and the transfer function vector of the nth layer can be defined as

$$\Psi^{\tau}(v)\equiv\begin{bmatrix}T_1(v) & T_2(v) & \cdots & T_{R^{\tau}}(v)\end{bmatrix}^T, \quad \tau=1,2,\cdots,S_j$$

where $T_l\ (v)\ (l=1,2,\cdots,R^{\tau})$ are the transfer functions associated with $\Psi^{\tau}(v)$. Then the final output of the jth isolated NN subsystem can be inferred as follows:

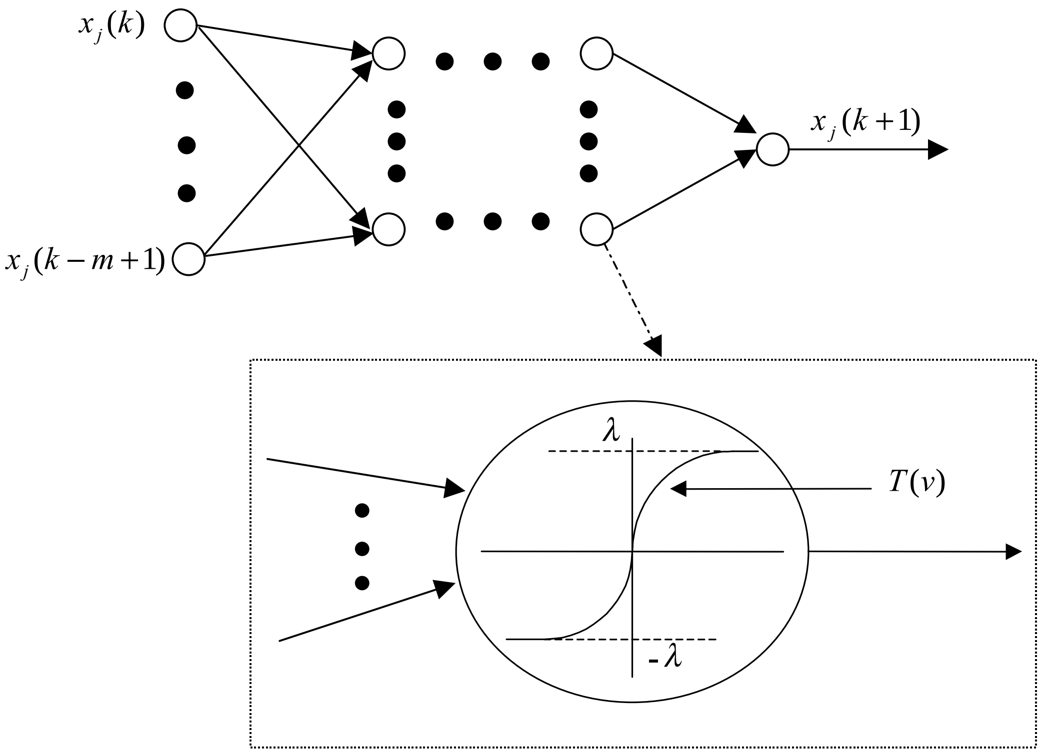

Fig. 1. The jth isolated NN subsystem.

$$x_j(k+1)=\Psi^{S_j}(W^{S_j}\Psi^{S_j-1}(W^{S_j-1}\Psi^{S_j-2}(\cdots\cdots\Psi^{2}(W^{2}\Psi^{1}(W^{1}X_j(k)))\cdots\cdots))), \tag{2}$$

where

$$X_j^{\,T}(k)=\begin{bmatrix}x_j(k) & x_j(k+1) & \cdots & x_j(k-m+1)\end{bmatrix}.$$

Next, a linear differential inclusion (LDI) system in the state-space representation is introduced and it can be described as follows [10] :

$$y(k+1) = A(z(k))y(k), \qquad A(z(k)) = \sum_{i=1}^{r_j} h_i(z(k))\overline{A}_i \,, \tag{3}$$

where r_j is a positive integer, $z(k)$ is a vector signifying the dependence of $h_i(\cdot)$ on its the elements and $y(k) = [y_1(k) \quad y_2(k) \quad \ldots \quad y_m(k)]^T$. Furthermore, it is assumed that

$$h_i(z(k)) \geq 0\,, \qquad \sum_{i=1}^{r_j} h_i(z(k)) = 1\,.$$

From the properties of LDI, without loss of generality, we can use $h_i(k)$ instead of $h_i(z(k))$. Then, the procedure of the conversion of the jth isolated NN subsystem (2) into an LDI representation is given as follows [9].

First, it can be found obviously that the output of transfer function $T(v)$ satisfies

$$\begin{aligned} g_1 v \leq T(v) \leq g_2 v, \qquad & v \geq 0 \\ g_2 v \leq T(v) \leq g_1 v, \qquad & v < 0 \end{aligned}$$

where g_1 and g_2 are the minimum value and the maximum value of the derivative of $T(v)$, and they are given in the following:

$$g(T) = \begin{cases} g_1 = \min\limits_{v} \dfrac{d\,T(v)}{dv} = 0 \\ g_2 = \max\limits_{v} \dfrac{d\,T(v)}{dv} = \dfrac{\lambda}{2\varsigma} \end{cases} \tag{4}$$

Subsequently, given transfer function vectors $\Psi^{\tau}(v)$ and net input vectors v^{τ}, the min-max matrix $G(v^{\tau}, \Psi^{\tau})$ is defined as follows:

$$G(v^{\tau}, \Psi^{\tau}) = diag(g(T_l))\,, \quad l = 1,\ 2,\ \ldots,\ R^{\tau}\,. \tag{5}$$

Moreover, based on the interpolation method and Eq. (2), we can obtain

$$\begin{aligned} x_j(k+1) &= [\sum_{q_1^{S_j}=1}^{2} \cdots \sum_{q_{R^{S_j}}^{S_j}=1}^{2} h_{q_1^{S_j}}^{S_j}(k) \cdots h_{q_{R^{S_j}}^{S_j}}^{S_j}(k) G(v^{S_j}, \Psi^{S_j})(W^{S_j}[\cdots[\sum_{q_1^{2}=1}^{2} \cdots \sum_{q_{R^2}^{2}=1}^{2} h_{q_1^2}^{2}(k) \cdots h_{q_{R^2}^2}^{2}(k) \\ &\qquad G(v^2, \Psi^2)(W^2[\sum_{q_1^{1}=1}^{2} \cdots \sum_{q_{R^1}^{1}=1}^{2} h_{q_1^1}^{1}(k) \cdots h_{q_{R^1}^1}^{1}(k) G(v^1, \Psi^1)(W^1 X_j(k))])]\cdots])] \\ &= \sum_{v^{S_j}} \cdots \sum_{v^1} h_{v^{S_j}}^{S_j}(k) \cdots h_{v^1}^{1}(k) G(v^{S_j}, \Psi^{S_j}) W^{S_j} \cdots G(v^1, \Psi^1) W^1 X_j(k) \\ &= \sum_{v_j} h_{v_j}(k) A_{v_j}(W, \Psi) X_j(k)\,, \end{aligned} \tag{6}$$

where

$$h_1^\tau(k), h_2^\tau(k) \in [0, \quad 1], \qquad \sum_{q_l^\tau=1}^{2} h_{q_l^\tau}^\tau(k) = 1,$$

$$\sum_{v^\tau} h_{v^\tau}^\tau(k) = \sum_{q_1^\tau=1}^{2} \cdots \sum_{q_{R^\tau}^\tau=1}^{2} h_{q_1^\tau}^\tau(k) \cdots h_{q_{R^\tau}^\tau}^\tau(k),$$

$$A_{v_j}(W, \Psi) \equiv G(v^{S_j}, \Psi^{S_j}) W^{S_j} \cdots G(v^1, \Psi^1) W^1,$$

$$\sum_{v_j} h_{v_j}(k) \equiv \sum_{v^{S_j}} \cdots \sum_{v^1} h_{v^{S_j}}^{S_j}(k) \cdots h_{v^1}^1(k) = 1,$$

$$h_{v_j}(k) \ge 0.$$

According to Eq. (3), the dynamics of the *j*th isolated NN subsystem (6) can be rewritten as the following LDI representation:

$$X_j(k+1) = \sum_{i=1}^{r_j} h_{ij}(k) \overline{A}_{ij} X_j(k), \tag{7}$$

where $\quad h_{ij}(k) \ge 0, \quad \sum_{i=1}^{r_j} h_{ij}(k) = 1,$

$\overline{A}_{ij}(k)$ is a constant matrix with appropriate dimension associated with $A_v(W, \Psi)$ and r_j is a positive integer.

Based on the analysis above, the *j*th NN subsystem N_j with interconnections can be described as follows:

$$N_j: \begin{cases} X_j(k+1) = \sum_{i=1}^{r_j} h_{ij}(k) \overline{A}_{ij} X_j(k) + \phi_j(k) & \text{(8a)} \\ \phi_j(k) = \sum_{\substack{n=1 \\ n \ne j}}^{J} C_{nj} X_n(k), & \text{(8b)} \end{cases}$$

where C_{nj} is the interconnection matrix between the *n*th and *j*th subsystems.

In the following, a stability criterion is proposed to guarantee the asymptotic stability of the NN large-scale system ***N***. Prior to examination of asymptotic stability, a useful concept is given below.

Lemma 1 [11] **:** For any matrices *A* and *B* with appropriate dimension, we have

$$A^T B + B^T A \le \kappa A^T A + \kappa^{-1} B^T B$$

where κ is a positive constant.

Theorem 1: The NN large-scale system ***N*** is asymptotically stable, if there exists positive constant κ is chosen to satisfy

$$\textbf{(I)}\quad \hat{\lambda}_{ij} = \lambda_M(Q_{ij}) + \alpha_{ijn} < 0, \qquad \text{for} \quad i = 1, 2, \cdots, r_j; \quad j = 1, 2, \cdots, J \tag{9a}$$

$$\hat{\lambda}_{ifj} = \lambda_M(Q_{ifj}) + 2\alpha_{ijn} < 0, \qquad \text{for} \quad i < f < r_j; \quad j = 1, 2, \cdots, J \tag{9b}$$

or

$$\textbf{(II)}\quad \begin{bmatrix} \hat{\lambda}_{1j} & 1/2\hat{\lambda}_{12j} & \cdots & 1/2\hat{\lambda}_{1r_jj} \\ 1/2\hat{\lambda}_{12j} & \hat{\lambda}_{2j} & \cdots & 1/2\hat{\lambda}_{2r_jj} \\ \vdots & \vdots & \vdots & \vdots \\ 1/2\hat{\lambda}_{1r_jj} & 1/2\hat{\lambda}_{2r_jj} & \cdots & \hat{\lambda}_{r_jj} \end{bmatrix} < 0 \qquad \text{for} \qquad j = 1, 2, \cdots, J \tag{10}$$

where

$$\alpha_{ijn} = \tilde{\lambda}_M(Q_{ijn}) + \eta_j, \quad \eta_j = \sum_{n \neq j}^{J} (J-1)\lambda_M(P_n)\left\| C_{jn} \right\|^2,$$

$$\tilde{\lambda}_M(Q_{ijn}) = \lambda_M(Q_{ijn}) + \sum_{n=1}^{J} \kappa^{-1} \frac{J-1}{J}, \quad Q_{ij} \equiv \overline{A}_{ij}^T P_j \overline{A}_{ij} - P_j,$$

$$Q_{ifj} \equiv \overline{A}_{ij}^T P_j \overline{A}_{fj} + \overline{A}_{fj}^T P_j \overline{A}_{ij} - 2P_j, \quad Q_{ijn} = \kappa \overline{A}_{ij}^T P_j C_{nj} C_{nj}^T P_j \overline{A}_{ij},$$

$$P_j = P_j^T > 0,$$

and $\lambda_M(Q_{ij})$, $\lambda_M(Q_{ifj})$ and $\lambda_M(Q_{ijn})$ denotes the maximum eigenvalues of the matrix Q_{ij} Q_{ifj} and Q_{ijn}, respectively.

III. Example

Consider a neural-network (NN) large-scale system composed of three NN subsystems which are described as follows.

Subsystem 1:

The first isolated NN subsystem with two layers where the hidden layer contains two neurons and the output layer is a single neuron is shown in Fig. 2. From this figure, we have†

† The symbol v_{bc}^{a} denotes the net input of the *b*th neuron of the *a*th layer in the *c*th subsystem.

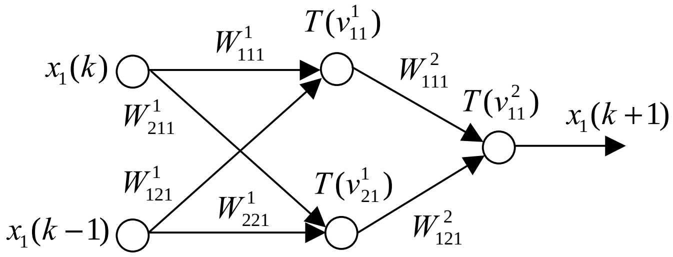

Fig. 2. The first isolated NN subsystem.

$$v^1_{l1} = W^1_{l11}x_1(k) + W^1_{l21}x_1(k-1), \qquad l=1, 2, \tag{11}$$

$$v^2_{11} = W^2_{111}T(v^1_{11}) + W^2_{121}T(v^1_{21}), \tag{12}$$

$$x_1(k+1) = T(v^2_{11}), \tag{13}$$

where

$$T(v^1_{l1}) = \frac{2}{1+\exp\left(-\frac{v^1_{l1}}{0.75}\right)} - 1, \quad l=1, 2, \tag{14}$$

$$T(v^2_{11}) = \frac{2}{1+\exp\left(-\frac{v^2_{11}}{0.75}\right)} - 1. \tag{15}$$

According to Eqs. (14-15), the minimum value and the maximum value of the derivative of $T(v)$ can be obtained as follows:

$$g_{11} = \min_v \frac{d\,T(v)}{dv} = 0, \quad g_{21} = \max_v \frac{d\,T(v)}{dv} = \frac{2}{3}. \tag{16}$$

Further, based on the interpolation method, Eqs. (14-15) can be represented, respectively,

$$T(v^1_{l1}) = (h^1_{l11}(k)g_{11} + h^1_{l21}(k)g_{21})v^1_{l1}, \quad l=1,2 \tag{17}$$

$$T(v^2_{11}) = (h^2_{111}(k)g_{11} + h^2_{121}(k)g_{21})v^2_{11}. \tag{18}$$

From Eqs. (13, 18), we have

$$x_1(k+1) = (h^2_{111}(k)g_{11} + h^2_{121}(k)g_{21})v^2_{11} = \sum_{i=1}^{2} h^2_{1i1}(k)g_{i1}v^2_{11}. \tag{19}$$

Substituting Eqs. (12, 16-17) into Eq. (19) yields

$$x_1(k+1) = \sum_{i=1}^{2} h_{1i1}^2(k) g_{i1} \sum_{p=1}^{2} W_{1p1}^2 T(v_{p1}^1) .$$

$$= \sum_{i=1}^{2} h_{1i1}^2(k) g_{i1} \sum_{p=1}^{2} W_{1p1}^2 \{h_{p11}^1(k) g_{11} + h_{p21}^1(k) g_{21}\} v_{p1}^1$$

$$= \sum_{i=1}^{2} h_{1i1}^2(k) g_{i1} \sum_{p=1}^{2}\sum_{s=1}^{2} h_{1p1}^1(k) h_{2s1}^1(k) \{g_{p1} W_{111}^2 v_{11}^1 + g_{s1} W_{121}^2 v_{21}^1\} . \tag{20}$$

By plugging Eq. (11) into Eq. (20), we obtain

$$x_1(k+1) = \sum_{i=1}^{2}\sum_{p=1}^{2}\sum_{s=1}^{2} h_{1i1}^2(k) h_{1p1}^1(k) h_{2s1}^1(k)$$

$$\cdot\{g_{i1}[g_{p1} W_{111}^2 W_{111}^1 + g_{s1} W_{121}^2 W_{211}^1] x_1(k) + g_{i1}[g_{p1} W_{111}^2 W_{211}^1 + g_{s1} W_{121}^2 W_{221}^1] x_1(k-1)\} \tag{21}$$

The matrix representation of Eq. (21) is

$$X_1(k+1) = \sum_{i=1}^{2}\sum_{p=1}^{2}\sum_{s=1}^{2} h_{1i1}^2(k) h_{1p1}^1(k) h_{2s1}^1(k) A_{ips} X_1(k) \tag{22}$$

where

$$A_{ips} = \begin{bmatrix} g_{i1}\{g_{p1} W_{111}^2 W_{111}^1 + g_{s1} W_{121}^2 W_{211}^1\} & g_{i1}\{g_{p1} W_{111}^2 W_{121}^1 + g_{s1} W_{121}^2 W_{221}^1\} \\ 1 & 0 \end{bmatrix}, \tag{23}$$

$$X_1^T(k) = \begin{bmatrix} x_1(k) & x_1(k-1) \end{bmatrix}.$$

Next, we assume that

$$W_{111}^1 = 1,\ W_{121}^1 = -0.5,\ W_{211}^1 = -1,\ W_{221}^1 = -0.5,\ W_{111}^2 = 1,\ W_{121}^2 = 1. \tag{24}$$

Substituting Eqs. (16, 24) into Eq. (23) yeilds

$$A_{111} = A_{112} = A_{121} = A_{122} = A_{211} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad A_{212} = \begin{bmatrix} -0.4444 & -0.2222 \\ 1 & 0 \end{bmatrix},$$

$$A_{221} = \begin{bmatrix} 0.4444 & -0.2222 \\ 1 & 0 \end{bmatrix}, \qquad A_{222} = \begin{bmatrix} 0 & -0.4444 \\ 1 & 0 \end{bmatrix}. \tag{25}$$

After all, by renumbering the matrices, the first isolated NN subsystem (22) can be rewritten as the following LDI representation:

$$X_1(k+1)=\sum_{i=1}^{4}h_{i1}(k)\overline{A}_{i1}X_1(k) \tag{26}$$

where

$$\overline{A}_{11}=A_{111}=A_{112}=A_{121}=A_{122}=A_{211},$$

$$\overline{A}_{21}=A_{212},\qquad \overline{A}_{31}=A_{221},\qquad \overline{A}_{41}=A_{222}, \tag{27}$$

$$\begin{aligned}h_{11}(k)&=h^{2}_{111}(k)h^{1}_{111}(k)h^{1}_{211}(k)+h^{2}_{111}(k)h^{1}_{111}(k)h^{1}_{221}(k)+h^{2}_{111}(k)h^{1}_{121}(k)h^{1}_{211}(k)\\&\quad+h^{2}_{111}(k)h^{1}_{121}(k)h^{1}_{221}(k)+h^{2}_{121}(k)h^{1}_{111}(k)h^{1}_{211}(k),\end{aligned}$$

$$h_{21}(k)=h^{2}_{121}(k)h^{1}_{111}(k)h^{1}_{221}(k),$$

$$h_{31}(k)=h^{2}_{121}(k)h^{1}_{121}(k)h^{1}_{211}(k),$$

$$h_{41}(k)=h^{2}_{121}(k)h^{1}_{121}(k)h^{1}_{221}(k).$$

Subsystem 2:

The second isolated NN subsystem with two layers where the hidden layer contains three neurons and the output layer is a single neuron is shown in Fig. 3. From this figure, we have

$$v^{1}_{l2}=W^{1}_{l12}x_2(k)+W^{1}_{l22}x_2(k-1),\qquad l=1,2,3 \tag{28}$$

$$v^{2}_{12}=W^{2}_{112}T(v^{1}_{12})+W^{2}_{122}T(v^{1}_{22})+W^{2}_{132}T(v^{1}_{32}), \tag{29}$$

$$x_2(k+1)=T(v^{2}_{12}), \tag{30}$$

where

$$T(v^{1}_{l2})=\frac{2}{1+\exp\left(-\frac{v^{1}_{l2}}{0.7}\right)}-1,\quad l=1,2,3 \tag{31}$$

$$T(v^{2}_{12})=\frac{2}{1+\exp\left(-\frac{v^{2}_{12}}{0.7}\right)}-1. \tag{32}$$

According to Eqs. (31-32), the minimum value and the maximum value of the derivative of $T(v)$ can be obtained as follows:

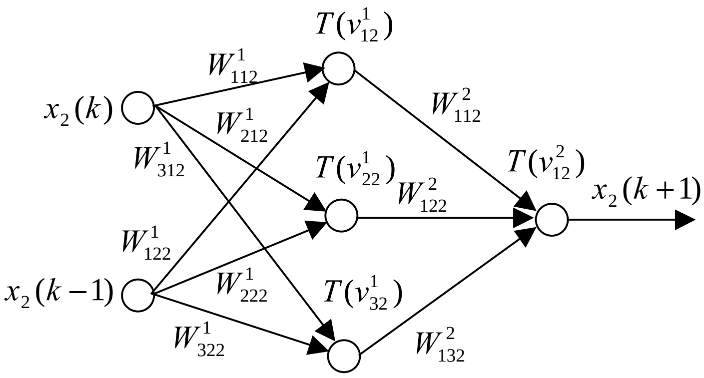

Fig. 3. The second isolated NN subsystem.

$$g_{12}=\min_v \frac{d\,T(v)}{dv}=0,\quad g_{22}=\max_v \frac{d\,T(v)}{dv}=\frac{5}{7}. \tag{33}$$

Using the same procedure as that in subsystem1, we obtain

$$X_2(k+1)=\sum_{i=1}^{2}\sum_{p=1}^{2}\sum_{s=1}^{2}\sum_{t=1}^{2} h_{1i2}^2(k)h_{1p2}^1(k)h_{2s2}^1(k)h_{3t2}^1(k)A_{ipst}X_2(k), \tag{34}$$

where

$$A_{ipst}=\begin{bmatrix} g_{i2}\{g_{p2}W_{112}^2W_{112}^1+g_{s2}W_{122}^2W_{212}^1+g_{t2}W_{132}^2W_{312}^1\} & g_{i2}\{g_{p2}W_{112}^2W_{122}^1+g_{s2}W_{122}^2W_{222}^1+g_{t2}W_{132}^2W_{322}^1\} \\ 1 & 0 \end{bmatrix}, \tag{35}$$

$X_2^{\,T}(k)=[x_2(k)\quad x_2(k-1)]$.

Next, we assume that

$$W_{112}^1=0.5,\ W_{212}^1=0.5,\ W_{312}^1=0.25,\ W_{122}^1=0.25,$$
$$W_{222}^1=0.35,\ W_{322}^1=0.5,\ W_{112}^2=0.25,\ W_{122}^2=-0.75,\ W_{132}^2=1, \tag{36}$$

Similarly, the second isolated NN subsystem (34) can be rewritten as the following LDI representation:

$$X_2(k+1)=\sum_{i=1}^{8} h_{i2}(k)\overline{A}_{i2}X_2(k) \tag{37}$$

where

$$\overline{A}_{12}=A_{1pst}=A_{2111}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\quad p, s, t=1, 2,\qquad \overline{A}_{22}=A_{2112}=\begin{bmatrix}0.1276 & 0.2552\\ 1 & 0\end{bmatrix},$$

$$\overline{A}_{32}=A_{2121}=\begin{bmatrix}-0.1913 & -0.1339\\ 1 & 0\end{bmatrix},\ \overline{A}_{42}=A_{2122}=\begin{bmatrix}-0.0638 & 0.1212\\ 1 & 0\end{bmatrix},\overline{A}_{52}=A_{2211}=\begin{bmatrix}0.0638 & 0.0319\\ 1 & 0\end{bmatrix},$$

$$\bar{A}_{62} = A_{2212} = \begin{bmatrix} 0.1913 & 0.2870 \\ 1 & 0 \end{bmatrix}, \quad \bar{A}_{72} = A_{2221} = \begin{bmatrix} -0.1276 & -0.1020 \\ 1 & 0 \end{bmatrix}, \quad \bar{A}_{82} = A_{2222} = \begin{bmatrix} 0 & 0.1531 \\ 1 & 0 \end{bmatrix}. \quad (38)$$

$$\begin{aligned} h_{12}(k) = {} & h_{112}^{2}(k)h_{112}^{1}(k)h_{212}^{1}(k)h_{312}^{1}(k) + h_{112}^{2}(k)h_{112}^{1}(k)h_{212}^{1}(k)h_{322}^{1}(k) \\ & + h_{112}^{2}(k)h_{112}^{1}(k)h_{222}^{1}(k)h_{312}^{1}(k) + h_{112}^{2}(k)h_{112}^{1}(k)h_{222}^{1}(k)h_{322}^{1}(k) \\ & + h_{112}^{2}(k)h_{122}^{1}(k)h_{212}^{1}(k)h_{312}^{1}(k) + h_{112}^{2}(k)h_{122}^{1}(k)h_{212}^{1}(k)h_{322}^{1}(k) \\ & + h_{112}^{2}(k)h_{122}^{1}(k)h_{222}^{1}(k)h_{312}^{1}(k) + h_{112}^{2}(k)h_{122}^{1}(k)h_{222}^{1}(k)h_{322}^{1}(k) \\ & + h_{122}^{2}(k)h_{112}^{1}(k)h_{212}^{1}(k)h_{312}^{1}(k), \end{aligned}$$

$$h_{22}(k) = h_{122}^{2}(k)h_{112}^{1}(k)h_{212}^{1}(k)h_{322}^{1}(k), \quad h_{32}(k) = h_{122}^{2}(k)h_{112}^{1}(k)h_{222}^{1}(k)h_{312}^{1}(k),$$

$$h_{42}(k) = h_{122}^{2}(k)h_{112}^{1}(k)h_{222}^{1}(k)h_{322}^{1}(k), \quad h_{52}(k) = h_{122}^{2}(k)h_{122}^{1}(k)h_{212}^{1}(k)h_{312}^{1}(k),$$

$$h_{62}(k) = h_{122}^{2}(k)h_{122}^{1}(k)h_{212}^{1}(k)h_{322}^{1}(k), \quad h_{72}(k) = h_{122}^{2}(k)h_{122}^{1}(k)h_{222}^{1}(k)h_{312}^{1}(k),$$

$$h_{82}(k) = h_{122}^{2}(k)h_{122}^{1}(k)h_{222}^{1}(k)h_{322}^{1}(k).$$

Subsystem 3:

The third isolated NN subsystem with two layers where the hidden layer contains two neurons and the output layer is a single neuron is shown in Fig. 4. From this figure, we have

$$v_{l3}^{1} = W_{l13}^{1} x_3(k) + W_{l23}^{1} x_3(k-1), \qquad l=1, 2, \tag{39}$$

$$v_{13}^{2} = W_{113}^{2} T(v_{13}^{1}) + W_{123}^{2} T(v_{23}^{1}), \tag{40}$$

$$x_3(k+1) = T(v_{13}^{2}), \tag{41}$$

where

$$T(v_{l3}^{1}) = \frac{2}{1+\exp\left(-\frac{v_{l3}^{1}}{0.6}\right)} - 1, \quad l=1, 2, \tag{42}$$

$$T(v_{13}^{2}) = \frac{2}{1+\exp\left(-\frac{v_{13}^{2}}{0.6}\right)} - 1. \tag{43}$$

According to Eqs. (42-43), the minimum value and the maximum value of the derivative of $T(v)$ can be obtained as follows:

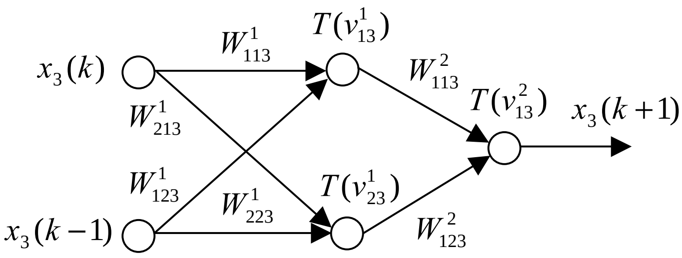

Fig. 4. The third isolated NN subsystem.

$$g_{13} = \min_v \frac{d\,T(v)}{dv} = 0, \quad g_{23} = \max_v \frac{d\,T(v)}{dv} = \frac{5}{6}. \tag{44}$$

Using the same procedure as that in subsystem1, we obtain

$$X_3(k+1) = \sum_{i=1}^{2}\sum_{p=1}^{2}\sum_{s=1}^{2} h_{1i3}^2(k) h_{1p3}^1(k) h_{2s3}^1(k) \overline{A}_{ips} X_3(k) \tag{45}$$

where

$$A_{ips} = \begin{bmatrix} g_{i3}\{g_{p3}W_{113}^2 W_{113}^1 + g_{s3}W_{123}^2 W_{213}^1\} & g_{i3}\{g_{p3}W_{113}^2 W_{123}^1 + g_{s3}W_{123}^2 W_{223}^1\} \\ 1 & 0 \end{bmatrix}, \tag{46}$$

$$X_3^T(k) = [x_3(k) \quad x_3(k-1)].$$

Next, assume that

$$W_{113}^1 = -0.5, \; W_{123}^1 = 1, \; W_{213}^1 = 0.25, \; W_{223}^1 = 0.2, \; W_{113}^2 = 0.5, \; W_{123}^2 = -1. \tag{47}$$

In similar fashion, the third isolated NN subsystem (45) can be rewritten as the following LDI representation:

$$X_3(k+1) = \sum_{i=1}^{4} h_{i3}(k) \overline{A}_{i3} X_3(k) \tag{48}$$

where

$$\overline{A}_{13} = A_{111} = A_{112} = A_{121} = A_{122} = A_{211} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; \overline{A}_{23} = A_{212} = \begin{bmatrix} -0.1736 & -0.1389 \\ 1 & 0 \end{bmatrix},$$

$$\overline{A}_{33} = A_{221} = \begin{bmatrix} -0.1736 & 0.3472 \\ 1 & 0 \end{bmatrix}, \quad \overline{A}_{43} = A_{222} = \begin{bmatrix} -0.3472 & 0.2083 \\ 1 & 0 \end{bmatrix}, \tag{49}$$

$$\begin{aligned} h_{13}(k) = {} & h_{113}^2(k)h_{113}^1(k)h_{213}^1(k) + h_{113}^2(k)h_{113}^1(k)h_{223}^1(k) + h_{113}^2(k)h_{123}^1(k)h_{213}^1(k) \\ & + h_{113}^2(k)h_{123}^1(k)h_{223}^1(k) + h_{123}^2(k)h_{113}^1(k)h_{213}^1(k), \end{aligned}$$

$h_{23}(k) = h_{123}^2(k)h_{113}^1(k)h_{223}^1(k)$,

$h_{33}(k) = h_{123}^2(k)h_{123}^1(k)h_{213}^1(k)$,

$h_{43}(k) = h_{123}^2(k)h_{123}^1(k)h_{223}^1(k)$.

Moreover, the interconnection matrices among three subsystems are in the following:

$$C_{12} = \begin{bmatrix} 0.1 & -0.15 \\ 0 & 0 \end{bmatrix}, \quad C_{13} = \begin{bmatrix} -0.16 & -0.13 \\ 0 & 0 \end{bmatrix}, \quad C_{21} = \begin{bmatrix} 0.13 & -0.12 \\ 0 & 0 \end{bmatrix},$$

$$C_{23} = \begin{bmatrix} -0.15 & 0.12 \\ 0 & 0 \end{bmatrix}, \quad C_{31} = \begin{bmatrix} 0.12 & -0.1 \\ 0 & 0 \end{bmatrix}, \quad C_{32} = \begin{bmatrix} 0.12 & 0.1 \\ 0 & 0 \end{bmatrix}.$$

Therefore, the NN large-scale system can be represented as follows:

$$N: \begin{cases} X_1(k+1) = \sum_{i=1}^{4} h_{i1}(k)\overline{A}_{i1}X_1(k) + \phi_1(k) & \text{(50a)} \\ X_2(k+1) = \sum_{i=1}^{8} h_{i2}(k)\overline{A}_{i2}X_2(k) + \phi_2(k) & \text{(50b)} \\ X_3(k+1) = \sum_{i=1}^{4} h_{i3}(k)\overline{A}_{i3}X_3(k) + \phi_3(k) & \text{(50c)} \\ \phi_j(k) = \sum_{\substack{n=1 \\ n\neq j}}^{3} C_{nj}X_n(k). & \text{(50d)} \end{cases}$$

In order to satisfy conditions (9), the matrix Q_{ij} in Eq. (10) must be chosen to be negative definite. Hence, based on Eqs. (25, 38, 49), we can obtain the following matrices P_j (j=1, 2, 3) by using LMI optimization algorithms such that Q_{ij}, $i = 1, 2, ..., r_j$; j=1, 2, 3 are negative definite with $\kappa = \frac{1}{7}$:

$$P_1 = \begin{bmatrix} 75.5477 & -0.1361 \\ -0.1361 & 33.3816 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 73.6939 & 1.2549 \\ 1.2549 & 38.7814 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 65.0683 & -0.9759 \\ -0.9759 & 35.7440 \end{bmatrix}. \qquad (51)$$

From Eq. (10), we have

$$\Lambda_1 = \begin{bmatrix} -24.0460 & -24.0455 & -24.0455 & -24.0450 \\ -24.0455 & -0.0348 & -9.0300 & -2.2617 \\ -24.0455 & -9.0300 & -0.6489 & -2.6686 \\ -24.0450 & -2.2617 & -2.6686 & -0.1526 \end{bmatrix},$$

$$\Lambda_2 = \begin{bmatrix} -25.1950 & -25.1280 & -25.3659 & -25.3096 & -25.1320 & -25.0621 & -25.3052 & -25.2458 \\ -25.1280 & -19.0678 & -21.6630 & -21.6611 & -20.9184 & -17.6653 & -21.6279 & -21.5593 \\ -25.3659 & -21.6630 & -21.1838 & -22.3407 & -23.6933 & -23.5595 & -22.0473 & -22.7927 \\ -25.3096 & -21.6611 & -22.3407 & -23.9360 & -24.8798 & -24.9905 & -24.0321 & -24.2421 \\ -25.1320 & -20.9184 & -23.6399 & -24.8798 & -24.8548 & -24.1769 & -25.3892 & -25.1597 \\ -25.0621 & -17.6635 & -23.5595 & -24.9950 & -24.1768 & -15.9050 & -20.6019 & -20.3412 \\ -25.3052 & -21.6279 & -22.0473 & -24.0321 & -25.3892 & -20.6019 & -22.5228 & -23.7997 \\ -25.2458 & -21.5593 & -22.7927 & -24.2421 & -25.1597 & -20.3412 & -20.3412 & -24.3967 \end{bmatrix},$$

$$\Lambda_3 = \begin{bmatrix} -19.8351 & -19.6494 & -19.7135 & -19.5301 \\ -19.6494 & -14.1933 & -15.0484 & -12.7872 \\ -19.7135 & -15.0484 & -7.3156 & -5.772 \\ -19.5301 & -12.7872 & -5.7721 & -1.9178 \end{bmatrix} \quad (52)$$

and the eigenvalues of them are given below:

$$\lambda(\Lambda_1) = -59.1586, -0.1000, 8.6833, 25.6930 \quad (53)$$

$$\lambda(\Lambda_2) = -186.0796, -2.9920, -2.3387, 0.1960, 2.7413, 3.0275, 8.4667 \quad (54)$$

$$\lambda(\Lambda_3) = -60.1354, 0.6209, 2.5494, 13.7033 . \quad (55)$$

Although the matrices Λ_j $(j=1,2,3)$ are not positive definite, the inequality (9) is satisfied. Therefore, based on condition (I) of Theorem 1, the NN large-scale system $\boldsymbol{N}$ is asymptotically stable. Simulation results of each subsystem are illustrated in Figs. 5-7 with initial conditions, $x_1(0) = -2$, $x_2(0) = -3$ and $x_3(0) = 2$.

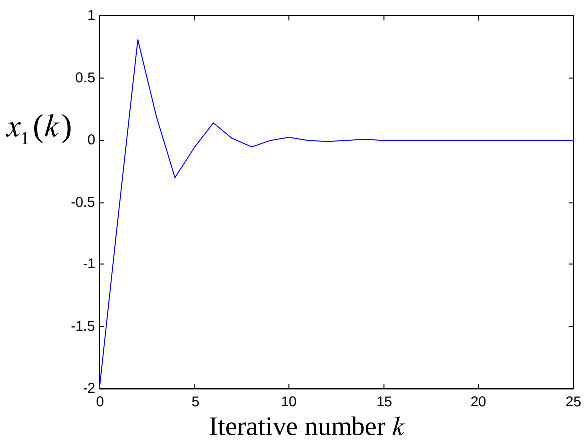

Fig. 5. The state $x_1(k)$ of subsystem 1.

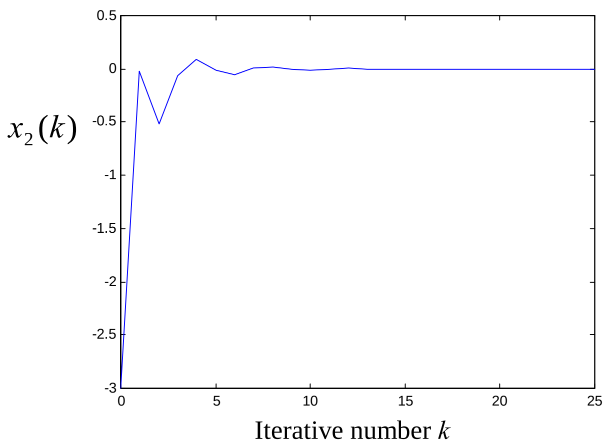

Fig. 5. The state $x_2(k)$ of subsystem 2.

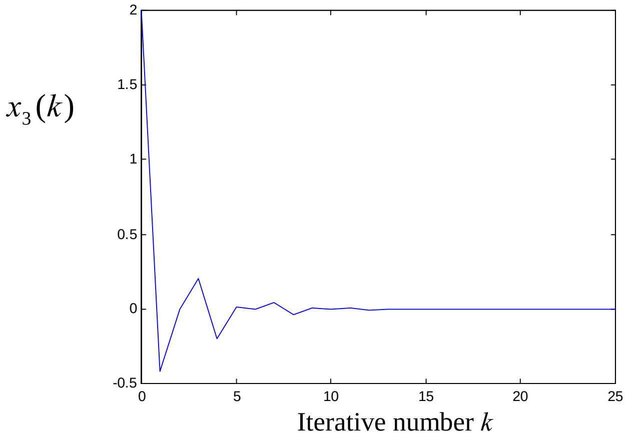

Fig. 5. The state $x_3(k)$ of subsystem 3.

Conclusions

In this paper, a stability criterion is derived to guarantee the asymptotic stability of neural-network (NN) large-scale systems. First, the dynamics of each NN model is converted into LDI (linear differential inclusion) representation. Subsequently, based on the LDI representations, the stability criterion in terms of Lyapunov's direct method is derived to guarantee the asymptotic stability of NN large-scale systems. Our approach is conceptually simple and straightforward. If the stability criterion is fulfilled, the NN large-scale system is asymptotically stable. Finally, a numerical example with simulations is given to illustrate the results.

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