1998 International Computer Symposium
Workshop on Algorithms
December 17-19, 1998, N.C.K.U., Tainan, Taiwan, R.O.C.

AN OPTIMAL ALGORITHM FOR SOLVING THE WEIGHTED MEDIAN PROBLEM ON THE WEIGHTED 4-CACTUS GRAPHS

Yu-Feng Lan and Yue-Li Wang

Department of Information Management,
National Taiwan University of Science and Technology, Taipei, Taiwan, R.O.C.
Email: ylwang@cs.ntust.edu.tw

ABSTRACT

In this paper, we solve the weighted median problem on a 4-cactus graph. The median problem has been extensively studied in the last three decades. On general graphs, the problem can be solved in $O(n^3)$, where n is the number of vertices in a graph. For tree graphs, however, more efficient algorithm can be devised to find the weighted median in $O(n)$. In this paper, we study weighted 4-cactus graphs which provide a less restrictive network structure than trees and show that the weighted median problem can be solved just as efficiently as on tree graphs.

1. INTRODUCTION

In this paper, we shall propose an optimal algorithm for finding the weighted medians of a 4-cactus graph. Let $G(V, E)$ be a finite, connected, undirected simple (i.e., no parallel edges and no self loops) graph. Let V, or $V(G)$, denote the vertex set of G and E, or $E(G)$, denote the edge set of G. The cardinality of a finite set X is denoted $|X|$. A *weighted graph* G is a graph in which a number $w_G(e)$ (respectively, $w_G(u)$) is associated with every edge $e \in E$ (respectively, vertex $u \in V$). We also use uv to denote edge e if u and v are the two incident vertices of e. The weight of e can also be represented by $w_G(u, v)$. The numbers $w_G(e)$ and $w_G(u)$ are called the weights of edge e and vertex u, respectively. A *path* of G is an alternating sequence of distinct vertices and edges, beginning and ending with vertices. The *length* of a path is the sum of the weights of the edges in the path. A path from u to v is a *shortest path* from u to v if there is no path from u to v with lower length. The distance from vertex u to vertex v, denoted $d_G(u, v)$, is the length of a shortest path from u to v. Note that $d_G(u, u) = 0$. The *w-distance* between vertices u and v, denoted $d_{G,w}(u, v)$, is $d_G(u, v) \times w_G(v)$. The sum of the w-distance from vertex u to every vertex of G is denoted by $D_G(u)$. A vertex u with the minimum $D_G(u)$ is called a *weighted median* of G [14].

A *subgraph* of $G(V, E)$ is a graph whose vertices and edges are subsets of V and E, respectively. A maximal connected subgraph of G is called a *component*. A *cut vertex* of G is a vertex whose removal increases the number of components. A maximal nonseparable subgraph is termed a *block* of $G(V, E)$. A *cycle* of G is a connected subgraph in which every vertex has exactly two distinct edges emanating from it. A cycle has exactly k vertices is called a *k-cycle*. A *cactus graph* (or *treelike graph*) is a connected graph in which every block with three or more vertices is a cycle [10]. A *4-cactus graph*, then, is a cactus graph whose cyclic blocks contain at most four vertices.

Example of treelike graphs exist for telecommunication networks, interstate highway systems, and computer communication networks. It was for telecommunication networks that

Hakimi originally proposed median and center objectives for the location of switching centers [8]. A simple feeder telecommunications network can be represented by a 3-cactus graph [12]. There are similar developments for the median and center problems on treelike graphs [3, 7, 10].

The one-median problem has been extensively studied in the literature and well documented in books by Francis et al. [5], Handler and Mirchandani [9], Minieka [16], and Mirchandani and Francis [17] and Korach et al. [11] and in the survey paper by Tansel et al. [19]. On tree networks, more efficient algorithms can be devised to find the medians. Goldman proposed an efficient algorithm to find the medians of a tree in linear time [6]. Recently, Peng and Lo presented an excellent method to solve the median problem on tree networks in linear time [18]. The weighted median of a general graph can be found in $O(|V|^3)$ time by using Floyd's algorithm which solves the all pairs shortest path problem [4]. Lee and Chang showed that the weighted median of a connected strongly chordal graph is a clique when all vertices are with positive weights [14]. In [10], Kincaid and Lowe gave a linear time algorithm to solve the absolute center problem on a 3-cactus graph. An absolute center is a point of a graph that minizes the maximum distance to all points of interest on the graph. Maimon and Kincaid also developed linear time algorithms to locate the vertex of minimum variance and the vertex of minimum average response time for any 3-cactus graph [15].

The rest of the paper is organized as follows: In Section 2, we describe some definition and terminology used throughout this paper. Some properties of tree and block graphs are also developed. In Section 3, we propose an optimal algorithm for the weighted median problem on a weighted 4-cactus graph. Finally, Section 4 contains the concluding remarks.

2. SOME PROPERTIES OF DISTANCE 4-BLOCK GRAPHS

In this section, we shall discuss some properties of distance 4-block graphs, which will be used to find the weighted medians of a 4-cactus graph. A *block graph* is a connected graph in which every block with three or more vertices is a clique [2]. A *distance block graph* is a block graph in which $w_G(e) = d_G(u, v)$ if $e = uv$ is an edge of G. A *distance 4-block graph*, then, is a distance block graph in which each block contains at most four vertices.

At first, we introduce some notation used in a rooted tree T whose root is r [1]. If a vertex v of T is adjacent to vertex u and u lies in the level below v, then u is called a *child* of v, and v is the *parent* of u. A vertex u is a *descendant* of v (and v is an *ancestor* of u) if the $v - u$ path in T lies below v. Let $ANC(u)$ denote the set of vertices which are on the path from u to r. Note that u itself is also in $ANC(u)$. The subtree of T rooted at vertex u is denoted by $T_r(u)$. The total weight of all vertices in subtree $T_r(u)$ is denoted by $W_T(u)$. Now, we state a relation between $D_T(r)$ and $D_T(u)$ as the following theorem.

Theorem 1 Let r be the root of T and u be a child of r. Then, $D_T(u) = D_T(r) + w_T(r, u) \times (W_T(r) - 2W_T(u))$.

Proof:

Q.E.D.

$$D_T(u) = \sum_{x\in T} d_T(u,x)\times w_T(x)$$
$$= \sum_{x\in T_r(u)} d_T(u,x)\times w_T(x) + \sum_{x\in T-T_r(u)} d_T(u,x)\times w_T(x)$$
$$= \sum_{x\in T_r(u)} (w_T(r,u)+d_T(u,x)-w_T(r,u))\times w_T(x) + \sum_{x\in T-T_r(u)} d_T(u,x)\times w_T(x)$$
$$= \sum_{x\in T_r(u)} (d_T(r,x)-w_T(r,u))\times w_T(x) + \sum_{x\in T-T_r(u)} d_T(u,x)\times w_T(x)$$
$$= \sum_{x\in T_r(u)} d_T(r,x)\times w_T(x) - \sum_{x\in T_r(u)} w_T(r,u)\times w_T(x) + \sum_{x\in T-T_r(u)} d_T(u,x)\times w_T(x)$$
$$= \sum_{x\in T_r(u)} d_T(r,x)\times w_T(x) - \sum_{x\in T_r(u)} w_T(r,u)\times w_T(x) + \sum_{x\in T-T_r(u)} (w_T(u,r)+d_T(r,x))\times w_T(x)$$
$$= \sum_{x\in T_r(u)} d_T(r,x)\times w_T(x) - \sum_{x\in T_r(u)} w_T(r,u)\times w_T(x) + \sum_{x\in T-T_r(u)} d_T(r,x)\times w_T(x) + \sum_{x\in T-T_r(u)} w_T(u,r)\times w_T(x)$$
$$= \sum_{x\in T} d_T(r,x)\times w_T(x) - \sum_{x\in T_r(u)} w_T(r,u)\times w_T(x) + \sum_{x\in T-T_r(u)} w_T(r,u)\times w_T(x)$$
$$= D_T(r) - \sum_{x\in T_r(u)} w_T(r,u)\times w_T(x) + \sum_{x\in T-T_r(u)} w_T(r,u)\times w_T(x)$$
$$= D_T(r) + w_T(r,u)(\sum_{x\in T-T_r(u)} w_T(x) - \sum_{x\in T_r(u)} w_T(x))$$
$$= D_T(r) + w_T(r,u)(\sum_{x\in T} w_T(x) - \sum_{x\in T_r(u)} w_T(x) - \sum_{x\in T_r(u)} w_T(x))$$
$$= D_T(r) + w_T(r,u)(\sum_{x\in T} w_T(x) - 2\sum_{x\in T_r(u)} w_T(x))$$
$$= D_T(r) + w_T(r,u)(W_T(r) - 2W_T(u)).$$

Breadth first traversal of the underlying graph of a connected weighted graph associates a spanning tree to the graph. It means that breadth first traversal partitions the edge of G into two types: *tree edges* (i.e., traversed edges) and *nontree edges* (also called *cross edges*). Let $G(V, E)$ be a distance 4-block graph, T be the breadth first spanning tree of G and $w_T(e) = w_G(e)$ (respectively, $w_T(u) = w_G(u)$) for every $e\in T$ (respectively, $u\in T$). In the breadth first traversal, the first visited vertex r of G is the root of T. We define the following notations with respect to T. A vertex v is called an *adjust vertex* of vertex u if v is an ancestor of u and v is adjacent with a cross edge. Let $A(u)$ denote the set of all adjust vertices of u. A cross edge e is said to be an *overpass* between vertices u and v, denoted $b(u, v)$, if e is the cross edge between a vertex in $ANC(u)$ and another vertex in $ANC(v)$. Vertex v is called an *overpass vertex* of vertex u if there exists an overpass between u and v. Let $B(u)$ denote the set of all overpass vertices of vertex u. Vertex u is called a *cross vertex* of vertex v if there exists a cross edge between u and v. Let $C(u)$ denote the set of all cross vertices of vertex u. Notice that a vertex has at most two cross vertices with respect to T. Let e be the cross edge between vertices u and v and x be the parent of u. Define that $\alpha(u, v) = w_G(x, u) + w_G(x, v) - w_G(e)$, $\beta(u) = \sum_{y\in C(u)}\alpha(u, y) \times W_T(y)$ and $\gamma(u) = \sum_{y\in A(u)}\beta(y)$. Note that $\alpha(u, v)$ can also be represented by $\alpha(e)$.

The following lemma states how to compute $d_G(u, v)$ from $d_T(u, v)$.

Lemma 1 Let G be a distance 4-block graph and T be a breadth first spanning tree of G. If vertex v is an overpass vertex of vertex u with respect to T, then $d_G(u, v) = d_T(u, v) - \alpha(b(u, v))$; otherwise $d_G(u, v) = d_T(u, v)$.

Proof:

Since G is a distance 4-block graph, it is clear that $d_G(u, v) = d_T(u, v)$ if v is not an overpass vertex of u. For the case where v is an overpass vertex of u, let vertices x and y be the two adjacent vertices of edge $b(u, v)$ and x and y are the ancestors of vertices u and v, respectively. Furthermore, let vertex z be the parent of vertex x. Then, $d_G(u, v)$ can be derived as follows:

$d_G(u, v) = d_T(u, x) + w_G(x, y) + d_T(y, v)$
$= d_T(u, x) + d_T(x, z) - d_T(x, z) + w_G(x, y) - d_T(z, y) + d_T(z, y) + d_T(y, v)$

$= d_T(u, z) - d_T(x, z) + w_G(x, y) - d_T(z, y) + d_T(z, v)$
$= d_T(u, v) - d_T(x, z) - d_T(z, y) + w_G(x, y)$
$= d_T(u, v) - \alpha(x, y)$
$= d_T(u, v) - \alpha(b(u, v))$.

Q.E.D.

The following theorem presents how to compute $D_G(u)$ from $D_T(u)$.

Theorem 2 Let $G(V, E)$ be a distance 4-block graph and T be its breadth first spanning tree whose root is r. Then, for any vertex $u \in V$, $D_G(u) = D_T(u) - \gamma(u)$.

Proof:

$$D_G(u) = \sum_{x \in V} d_G(u,x) \times w_G(x)$$
$$= \sum_{x \in V-B(u)} d_G(u,x) \times w_G(x) + \sum_{x \in B(u)} d_G(u,x) \times w_G(x)$$
$$= \sum_{x \in V-B(u)} d_T(u,x) \times w_G(x) + \sum_{x \in B(u)} d_G(u,x) \times w_G(x)$$
$$= \sum_{x \in V-B(u)} d_T(u,x) \times w_G(x) + \sum_{x \in B(u)} (d_T(u,x) - \alpha(b(u,x))) \times w_T(x)$$
$$= \sum_{x \in V-B(u)} d_T(u,x) \times w_G(x) + \sum_{x \in B(u)} d_T(u,x) \times w_T(x) - \sum_{x \in B(u)} \alpha(b(u,x)) \times w_T(x)$$
$$= \sum_{x \in V} d_T(u,x) \times w_G(x) - \sum_{x \in B(u)} \alpha(b(u,x)) \times w_T(x)$$
$$= D_T(u) - \sum_{v \in A(u)} \sum_{y \in C(v)} \sum_{x \in T_r(y)} \alpha(v,y) \times w_T(x)$$
$$= D_T(u) - \sum_{v \in A(u)} \sum_{y \in C(v)} \alpha(v,y) \times W_T(y)$$
$$= D_T(u) - \sum_{v \in A(u)} \beta(v)$$
$$= D_T(u) - \gamma(u).$$

Q.E.D.

3. FINDING THE WEIGHTED MEDIANS OF A WEIGHTED 4-CACTUS GRAPH

Given a weighted 4-cactus graph $H(V, E)$, we can construct a corresponding distance 4-block graph $G(V, E')$ from H. The construction of G from H is described as follows. All of the edges in H are also in G and for each 4-cycle of H, two edges are inserted in order to form a block. The resulting graph is a 4-block graph $G(V, E')$. Moreover, $w_G(u) = w_H(u)$ for every vertex $u \in V$ and $w_G(e) = d_H(u, v)$ if $e = uv$ is an edge of E'. Now, we are at the position to describe our algorithm for finding the weighted medians of a weighted 4-cactus graph.

Algorithm A

Input: A weighted 4-cactus graph $H(V, E)$

Output: The weighted medians of H.

Method:

Step 1. Construct the corresponding distance 4-block graph $G(V, E')$ from H.

Step 2. Construct a breadth first spanning tree T from G. Let the root of T be r. Then, compute $D_T(r)$.

Step 3. Compute $W_T(u)$ and $\gamma(u)$ for each vertex $u \in T$.

Step 4. By using Theorem 1, compute $D_T(u)$ for each vertex $u \in T$ in preorder.

Step 5. By using Theorem 2, compute $D_G(u)$ for each vertex $u \in G$.

Step 6. The vertices u with the smallest value $D_G(u)$ are the weighted medians of H.

End of Algorithm

Since each cycle of a 4-cactus graph has at most four vertices, Step 1 takes $O(|V|)$ time to construct the corresponding distance 4-block graph $G(V, E')$ of a weighted 4-cactus graph $H(V,$

E). Note that $|E'| \leq 3|E| / 2$ since every 4-cycle needs two more edges to form a block. Constructing a breadth first spanning tree takes $O(|V| + |E'|)$ time [1]. Computing $D_T(u)$ also takes $O(|V| + |E'|)$ time. Thus, Step 2 needs $O(|V| + |E'|)$ time. Step 3 takes at most $O(|V| + |E'|)$ time to compute $W_T(u)$ and $\gamma(u)$ for all vertices u in T. Steps 4, 5 and 6 can be done in $O(|V|)$ time. Therefore, *Algorithm A* takes $O(|V| + |E|)$ time.

We conclude our result as the following theorem.

Theorem 3 Let $H(V, E)$ be a weighted 4-cactus graph. The weighted medians of H can be found in $O(|V|)$ time.

Proof:

Let G be the corresponding distance 4-block graph of $H(V, E)$. It is obvious that $d_H(u, v) = d_G(u, v)$ for every pair of vertices $u, v \in V$. Thus, $D_H(u) = D_G(u)$ for every vertex $u \in H$. According to the analysis of *Algorithm A*, the weighted medians of H can be found in $O(|V| + |E|)$ time. Since H is a planar graph, $O(|E|) = O(|V|)$ [13]. Therefore, the weighted medians of H can be found in $O(|V|)$ time. This completes the proof.

Q.E.D.

4. CONCLUDING REMARKS

In this paper, we propose an optimal algorithm for solving the weighted median problem on a weighted 4-cactus graph. On a general graph, the running time for finding the weighted median is $O(|V|^3)$. For special graph such as tree graphs, more efficient algorithm can be done in $O(|V|)$. The weighted 4-cactus graphs, which are more general than tree graphs, have many interesting properties. By taking advantage of the properties, we solve the weighted median problem on the weighted 4-block graphs and weighted 4-cactus graphs in linear time. Possible further research includes the application of our technique to other network problems.

REFERENCES

[1] V. Aho, J. E. Hopcroft, and J. D. Ullman, *The Design and Analysis of Computer Algorithms*, Addison-Wesley, Reading, MA, 1974.

[2] F. Buckley and F. Harary, *Distance in Graph*, Addison-Wesley, Reading, MA, 1990.

[3] M. L. Chen, R. L. Francis, J. F. Lawrence, T. J. Lowe and S. Tufekci, "Block-vertex Duality and the One-median Problem". *Networks* Vol. 15, 1986, pp. 395-412.

[4] R. W. Floyd, "Algorithm 97:Shortest Paths", *Communications of the ACM*, Vol. 5, 1962, pp. 345.

[5] R. L. Francis, L. F. McGinnis, and J. A. White, *Facility Layout and Locations: An Analytical Approach.* Prentice Hall, Englewood Cliffs, NJ 1992.

[6] J. Goldman, "Optimum center location in Simple Networks", *Transportation Science*, Vol. 5, 1971, pp. 212-221.

[7] Y. Gurevich, L. Stockmeyer and U. Vishkin, "Solving NP-hard Problems on Graphs That Are Almost Trees and an Application to Facility Location", *Journal of the Association of Computer Machinery*, Vol. 31, 1984, pp. 459-473.

[8] S. L. Hakimi, "Optimum Location of Switching Centers and the Absolute Centers and Medians of a Graph", *Operation Research*, Vol. 12, 1964, pp. 450-459.

[9] G. Y. Handler and P. B. Mirchandani, *Location on Networks: Theory and Algorithms* MIT Press, Cambridge, MA, 1979.

[10] R. K. Kincaid and T. J. Lowe, "Locating an Absolute Center on Graphs That Are Almost Trees", *European Journal of Operational Research*, Vol. 44, 1990, pp. 357-372.

[11] E. Korach, D. Rotem and N. Santoro, "Distributed Algorithms for Finding Centers and Medians in Networks", *ACM transactions on programming languages and systems*, Vol. 6, 1984, pp. 390-401.

[12] W. L. G. Koontz, "Economic Evaluation of Loop Feeder Relief Alternatives", *The Bell System Technical Journal*, Vol. 59, 1980, pp. 277-281.

[13] K. Kuratowski, "Sur le probleme des courbes gauches en topologie", *Fundamental Mathematics*, Vol. 15, 1930, pp. 271-283.

[14] Hai-Yen Lee and Gerard J. Chang, "The W-median of a Connected Strongly Chordal Graph", *Journal of Graph Theory*, Vol. 18, No. 7, 1994, pp. 673-680.

[15] O. Maimon and R. K. Kincaid, "Material Handling System Design for Flexible Manufacturing Systems: A Network Approach", *Working paper*, Department of Mathematics, College of William and Mary, Williamsburg, VA 23185.

[16] E. Minieka, *Optimization Algorithms for Networks and Graphs.* Marcel Dekker, New York 1978.

[17] P. B. Mirchandani and R. L. Francis, *Discrete Location Theory.* Wiley, New York 1990.

[18] Peng and W. Lo, "Efficient Algorithms for Finding a Core of a Tree with a Specified Length", *Journal of Algorithms*, Vol. 20, 1996, pp. 445-458.

[19] C. Tansel, R. L. Francis, and T. J. Lowe, "Location on Networks: A survey", *Management Science*, Vol. 29, 1983, pp. 482-511.