1998 International Computer Symposium
Workshop on Algorithms
December 17-19, 1998, N.C.K.U., Tainan, Taiwan, R.O.C.

Edge Domination of Distance-Hereditary Graphs

Maw-Shang Chang and Rue-Lin Yang

Department of Computer Science and Information Engineering,
National Chung Cheng University, Chiayi, Taiwan, R.O.C.
Email: mschang@cs.ccu.edu.tw

ABSTRACT

An edge dominating set D of a graph $G = (V, E)$ is a set of edges such that every edge not in D is adjacent to at least one edge in D. This paper gives an $O(n^4)$ time algorithm which finds a minimum edge dominating set for a distance-hereditary graph where $n = |V|$.

1. INTRODUCTION

In this paper, we consider only finite, undirected simple graphs. Given a graph $G = (V, E)$, two vertices u and v of G connected by an edge are called *adjacent vertices* and we denote the edge by (u, v). Two edges having a vertex in common or a vertex and its incident edge are also called *adjacent*. If two vertices or two edges are adjacent, one may say that they *dominate* each other. We also say that vertex v and edge e cover each other if they are adjacent. A *matching* of G is a set M of edges in which no two edges are adjacent. A maximum matching of G is a matching of maximum cardinality. The *maximum matching problem* involves finding a maximum matching for the given graph. A *vertex cover* of G is a set C of vertices such that every edge of G is covered by at least one vertex in C. A minimum vertex cover of G is a vertex cover of minimum cardinality. The *vertex cover problem* involves finding a minimum vertex cover for the given graph. An *edge dominating set* of G is a set D of edges such that every edge not in D is adjacent to at least one edge in D. An edge dominating set D' is independent if D' is also a matching of G. D' is also called *an independent* edge dominating set of G. A *minimum edge dominating set* of G is an edge dominating set of minimum cardinality. A minimum independent edge dominating set of G is an independent edge dominating set of minimum cardinality. Specifically, a minimum independent edge dominating set of G is a minimum maximal matching of G. The edge domination problem has been studied by several researchers [2, 3, 5, 7, 9, 10, 11, 12]. It involves finding a minimum edge dominating set for the given graph. Yannakakis and Gavril [2] showed that the size of a minimum edge dominating set is equal to the size of a minimum independent edge dominating set. Generally speaking, finding a minimum independent edge dominating set is easier than finding a minimum edge dominating set. Therefore, we will focus on finding a minimum independent edge dominating set of G. The distance $d_G(u, v)$ between two vertices u and v of a connected graph G is the minimum length of a u-v path in G. A graph is distance-hereditary if each pair of vertices are equidistant in every connected induced subgraph containing them. Howorka [1] firstly introduced the class of distance-hereditary graphs and studied characterizations of them. Bandelt and Mulder [4] gave a constructive characterization of distance-hereditary graphs which is called one vertex extensions. Hammer and Maffray [6] proposed a linear time recognition algorithm constructing a one vertex extension ordering of the given distance-hereditary graph. Chang et. al. [13] defined the one-vertex-extension tree and gave a new recursive definition for distance-hereditary graphs. They are useful for solving problems on distance-hereditary graphs. For solving the edge domination problem on distance-hereditary graphs, we solve a more general edge domination problem called the *mixed edge domination problem* where the edge domination problem is a special case of this problem. Then by using dynamic programming techniques based upon the new recursive definition of distance-hereditary graphs given by Chang et. al., we give an $O(n^4)$ time algorithm for solving the edge domination problem on distance-hereditary graphs.

2. PRELIMINARIES

We follow the notation and definitions given by Chang et. al. [13]. For a graph $G = (V, E)$ and a subset X of V, define $G[X]$ to be the subgraph of G induced by X. We represent the edge set of $G[X]$ by $E(G[X])$.

Definition 2.1 For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we refer to the graph $G = (V_1 \cup V_2, E_1 \cup E_2)$ as the *union* of graphs G_1 and G_2 (denoted by $G = G_1 \cup G_2$).

Definition 2.2 We say that two disjoint vertex subsets V_1 and V_2 of V form a *join* in graph $G = (V, E)$, if every vertex of V_1 is adjacent to all vertices of V_2.

Definition 2.3 (Chang et. al. [13]) The class of distance-hereditary graphs and the *twin set*, denoted by $TS(G)$ of a distance-hereditary graph G can be defined as follows.

- K_1 is a distance-hereditary graph and the twin set $TS(K_1)$ of K_1 is the only vertex of K_1.
- If G_1 and G_2 are distance-hereditary graphs, then the graph $G = G_1 \cup G_2$ is also a distance-hereditary graph. The twin set $TS(G)$ of G is the union of the twin sets of G_1 and G_2. In this case, we say that the graph G is formed from G_1 and G_2 by a *false twin* operation.
- If G_1 and G_2 are distance-hereditary graphs, then the

graph G obtained from G_1 and G_2 by connecting every vertex of the twin set of G_1 to all vertices of the twin set of G_2 is also a distance-hereditary graph. That is, $TS(G_1)$ and $TS(G_2)$ form a join in graph G. The twin set of G is the union of the twin sets of G_1 and G_2. In this case, we say that the graph G is formed from G_1 and G_2 by a *true twin* operation.

- If G_1 and G_2 are distance-hereditary graphs, then the graph G obtained from G_1 and G_2 by connecting every vertex of the twin set of G_1 to all vertices of the twin set of G_2 is also a distance-hereditary graph. That is, $TS(G_1)$ and $TS(G_2)$ form a join in graph G. The twin set of G is the twin set of G_1. In this case, we say that the graph G is formed from G_1 and G_2 by a *pendant vertex* operation. We also say that graph G is obtained from G_1 and G_2 by attaching graph G_2 to graph G_1.

3. AN $O(n^4)$ TIME ALGORITHM

Given a distance-hereditary graph $G = (V, E)$, a subset D of $V \cup E$ is called a mixed set if every edge in D is not adjacent to any vertex or edge in D and every vertex in D is in $TS(G)$. A *mixed set* is a matching if it does not contain any vertex in V. We denote $D \cap V$ and $D \cap E$ by $V(D)$ and $E(D)$, respectively. By $P(D)$, we denote the set of vertices in $V(D)$ and vertices to which the edges in D are incident. A *mixed edge dominating set* (MEDS) of G is a mixed set D such that every edge of G not in D is dominated by an edge in D or covered by a vertex in D. In particular, if an MEDS D of G satisfying $TS(G) \subseteq P(D)$, then we call it a *simple mixed edge dominating set* (SMEDS) of G. A minimum MEDS of G is an MEDS of minimum cardinality. Correspondingly, a minimum SMEDS of G is an SMEDS of minimum cardinality. The *mixed edge domination problem* involves finding a minimum MEDS of G. We need more notation and definitions for clarity.

Definition 3.1 Let $G = (V, E)$ be a distance-hereditary graph. A p-vertices MEDS D of G is an MEDS of G such that $|V(D)| = p$. A minimum p-vertices MEDS of G is a p-vertices MEDS of G of minimum cardinality. We denote a minimum p-vertices MEDS of G by $F(G, p)$ and $|F(G, p)|$ by $f(G, p)$, respectively.

Definition 3.2 A minimum p-vertices SMEDS of G is a p-vertices SMEDS of G of minimum cardinality. We denote a minimum p-vertices SMEDS of G by $F_T(G, p)$ and $|F_T(G, p)|$ by $f_T(G, p)$, respectively.

Obviously, a p-vertices MEDS of G is an edge dominating set of G if $p = 0$. If $p = 0$ then a p-vertices MEDS of G is also a matching. Yannakakis and Gavril showed that there exists a minimum edge dominating set which is also a matching [2]. Therefore, a minimum p-vertices MEDS of G with $p = 0$ is a minimum edge dominating set of G. So, we can find a minimum edge dominating set of a distance-hereditary graph G by finding a minimum p-vertices MEDS of G with $p = 0$. The following two lemmas play important roles in solving the edge domination problem on distance-hereditary graphs.

Lemma 3.1 Suppose $G = (V, E)$ is formed from $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ by a true twin operation or a pendant vertex operation. Let D be an MEDS of G. Then, there exists an MEDS D' of G satisfying that

(i) $V(D) = V(D')$,

(ii) $P(D) = P(D')$,

(iii) $|D| = |D'|$, and

(iv) $D' \cap E(G[TS(G_1)]) = \phi$ or $D' \cap E(G[TS(G_2)]) = \phi$.

Proof. We prove the lemma by contradiction. Let D' be an MEDS satisfying that (i) $V(D') = V(D)$, (ii) $P(D') = P(D)$, (iii) $|D'| = |D|$, and (iv) $|D' \cap (E(G[TS(G_1)]) \cup E(G[TS(G_2)]))|$ is minimum. Suppose D' contains edges from both $E(G[TS(G_1)])$ and $E(G[TS(G_2)])$. That is, $E(D')$ has an edges $e_1 = (u, v)$ from $E(G[TS(G_1)])$ and another edge $e_2 = (s, t)$ from $E(G[TS(G_2)])$. By definition, (u, s) and (v, t) are edges of G. Let $W = (D' - \{e_1, e_2\}) \cup \{(u, s), (v, t)\}$. Since $\{(u, s), (v, t)\} \subseteq E$ and $P(W) = P(D')$, we have that W is also an MEDS of G. Clearly $V(W) = V(D')$, $P(W) = P(D')$, $|W| = |D'|$ and W has less edges from $E(G[TS(G_1)]) \cup E(G[TS(G_2)])$ than D'. This contradicts the assumption that $|D' \cap (E(G[TS(G_1)]) \cup E(G[TS(G_2)]))|$ is minimum.

Q.E.D.

Lemma 3.2 Suppose $G = (V, E)$ is formed from $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ by a true twin operation or a pendant vertex operation. Let D be an MEDS of G. Then either $TS(G_1)$ or $TS(G_2)$ is a subset of $P(D)$.

Proof. Suppose neither $TS(G_1)$ nor $TS(G_2)$ is a subset of $P(D)$. Then there exist two vertices $u \in TS(G_1)$ and $V \in TS(G_2)$ with $\{u, v\} \cap P(D) = \phi$. By definition, $(u, v) \in E$. Obviously, D does not dominate the edge $(u, v) \in E$. This contradicts the assumption that D is an MEDS set of G. Consequently, the lemma is correct.

Q.E.D

Note that a p-vertices MEDS D of G with $TS(G) \subseteq P(D)$ is also a p-vertices SMEDS of G. For technical reasons, we let $f(G, p) = \infty$ if G has no p-vertices MEDS. Similarly, we let $f_T(G, p) = \infty$ if $F_T(G, p)$ does not exist. For example, if $p > |TS(G)|$, then $f_T(G, p) = \infty$ and $F_T(G, p)$ does not exist. We now show how to compute $f(G, p)$ and $f_T(G, p)$ for a distance-hereditary graph $G = (V, E)$ as follows. Moreover, we can find $F(G, p)$ and $F_T(G, p)$ when we compute $f(G, p)$ and $f_T(G, p)$. It is easy to see that (i) $F(G, p) = \phi$ and $F_T(G, p) = \phi$, $f(G, p) = 0$, and $f_T(G, p) = 0$ if $G = K_1$ and $p = 0$; (ii) $F(G, p) = V$ and $F_T(G, p) = V$ if $G = K_1$ and $p = 1$; and (iii) both $F(G, p)$ and $F_T(G, p)$ do not exist if $G = K_1$ and $p > 1$. If G is not K_1, then G is formed from two induced subgraphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ by one of three operations given in Definition 2.3. So we consider the following three cases. For simplicity, .let $|TS(G_1)| = n_1$ and $|TS(G_2)| = n_2$.

Case 1. G is formed from G_1 and G_2 by a false twin operation. In other words, $G = G_1 \cup G_2$.Let $D = F(G, p)$ be a minimum p-vertices MEDS of G. Assume that i vertices of

$V(D)$ are from $TS(G_1)$ and the other p-i vertices of $V(D)$ are from $TS(G_2)$. Let $P_1 = D \cap TS(G_1)$, $P_2 = D \cap TS(G_2)$, $M_1 = D \cap E_1$, and $M_2 = D \cap E_2$. Clearly, $|P_1| = i$ and $|P_2| = p$-i. It is not hard to see that $M_1 \cup P_1$ is a minimum i-vertices MEDS of G_1 and $M_2 \cup P_2$ is a minimum (p-i)-vertices MEDS of G_2. Thus,

$$f(G,p) = \min_{0 \le i \le q} \{f(G_1,i) + f(G_2 p - i)\}$$

where $q = \min\{p, n_1\}$. Correspondingly, let $S = F_T(G, p)$ be a minimum p-vertices SMEDS of G. Suppose that i vertices of $V(S)$ are from $TS(G_1)$ and the other p-i vertices of $V(S)$ are from $TS(G_2)$. Let $P_1 = S \cap TS(G_1)$, $P_2 = S \cap TS(G_2)$, $M_1 = S \cap E_1$, and $M_2 = S \cap E_2$ where $|P_1| = i$ and $|P_2| = p - i$. Obviously, $M_1 \cup P_1$ is a minimum i-vertices SMEDS of G_1 and $M_2 \cup P_2$ is a minimum (p-i)-vertices SMEDS of G_2. Thus,

$$f_T(G,p) = \min_{0 \le i \le q} \{f_T(G_1,i) + f_T(G_2,p-i)\}$$

where $q = \min\{p, n_1\}$.

Case 2. G is formed from G_1 and G_2 by a true twin operation. Let D be an $F(G, p)$. By Lemma 3.2, we know that either $TS(G_1) \subseteq P(D)$ or $TS(G_2) \subseteq P(D)$. First we consider how to compute $f(G, p)$ in this case. Suppose $TS(G_1) \subseteq P(D)$. Note that $|V(D)| = p$, $|D| = f(G, p)$, and $TS(G_1) \subseteq P(D)$. Since every vertex of $TS(G_1)$ connects to all vertices of $TS(G_2)$ and $TS(G_1) \subseteq P(D)$, the edges between $TS(G_1)$ and $TS(G_2)$ are covered by $TS(G_1)$. Assume that $P_1 = V(D) \cap TS(G_1)$, $P_2 = V(D) \cap TS(G_2)$, $M_1 = E(D) \cap E_1$, $M_2 = E(D) \cap E_2$, and $J = E(D) \cap (E - (E_1 \cup E_2))$. By definition, every edge of $E - (E_1 \cup E_2)$ connects a vertex in $TS(G_1)$ and a vertex in $TS(G_2)$. Thus every edge of J connects a vertex in $TS(G_1)$ and another vertex in $TS(G_2)$. Assume that $Q_1 = P(J) \cap TS(G_1)$, $Q_2 = P(J) \cap TS(G_2)$, $|P_1| = i$, $|P_2| = p$-i, and $|J| = k$. It is straightforward to verify that $P_1 \cup Q_1 \cup M_1$ is a minimum (i+k)-vertices SMEDS of G_1 and $P_2 \cup Q_2 \cup M_2$ is a minimum (p-i+k)-vertices MEDS of G_2. Thus, in this case we have that

$$f(G,p) = \min_{0 \le i \le q, 0 \le k \le h} \{f_T(G_1,i+k) + f(G_2,(p-i)+k) - k\}$$

where $q = \min\{p, n_1\}$ and $h = \min\{n_1 - i, n_2 - (p-i)\}$. On the other hand, suppose $TS(G_2) \subseteq P(D)$. By similar arguments, in this case we have that

$$f(G,p) = \min_{0 \le i \le q, 0 \le k \le h} \{f(G_1,i+k) + f_T(G_2,(p-i)+k) - k\}$$

where $q = \min\{p, n_1\}$ and $h = \min\{n_1 - i, n_2 - (p-i)\}$. Therefore we have that $f(G, p) = \min\{f_1, f_2\}$ where

$$f_1 = \min_{0 \le i \le q, 0 \le k \le h} \{f_T(G_1,i+k) + f(G_2,(p-i)+k) - k\}$$

and

$$f_2 = \min_{0 \le i \le q, 0 \le k \le h} \{f(G_1,i+k) + f_T(G_2,(p-i)+k) - k\}$$

where $q = \min\{p, n_1\}$ and $h = \min\{n_1 - i, n_2 - (p-i)\}$. Next we consider how to compute $f_T(G, p)$. By arguments similar to those for obtaining recursive formula for computing $f(G, p)$, we have that

$$f_T(G,p) = \min_{0 \le i \le q, 0 \le k \le h} \{f_T(G_1,i+k) + f_T(G_2,(p-i)+k) - k\}$$

where $q = \min\{p, n_1\}$ and $h = \min\{n_1 - i, n_2 - (p-i)\}$.

Case 3. G is formed from G_1 and G_2 by attaching G_2 to G_1. Let D be an $F(G, p)$. By Lemma 3.2, we know that either $TS(G_1) \subseteq P(D)$ or $TS(G_2) \subseteq P(D)$. This case is similar to Case 2. The difference between Case 2 and Case 3 is that an MEDS of G does not contain any vertex of $TS(G_2)$ if G is formed from G_1 and G_2 by attaching G_2 to G_1 but it may contain vertices from both $TS(G_1)$ and $TS(G_2)$ if G is formed from G_1 and G_2 by a true twin operation. In other words, $V(D) \subseteq TS(G_1)$ in Case 3. It is easy to see that $F(G, p)$ and $F_T(G, p)$ do not exist if $p > n_1$. In the following we assume that $n_1 \ge p$. By arguments similar to those for Case 2, we have that $f(G, p) = \min\{f_1, f_2\}$ where

$$f_1 = \min_{0 \le k \le h} \{f_T(G_1,p+k) + f(G_2,k) - k\}$$

and

$$f_2 = \min_{0 \le k \le h} \{f(G_1,p+k) + f_T(G_2,k) - k\}$$

where $h = \min\{n_1 - p, n_2\}$. Similarly, we have that

$$f_T(G,p) = \min_{0 \le k \le h} \{f_T(G_1,p+k) + f(G_2,k) - k\}$$

where $h = \min\{n_1 - p, n_2\}$.

With the above discussions, we have the following results.

Lemma 3.3 Given $f(G_1, p)$, $f_T(G_1, p)$, $f(G_2, p)$, and $f_T(G_2, p)$ for $n \ge p \ge 0$, it takes at most $O(n^3)$ time to compute $f(G, p)$ and $f_T(G, p)$ for $n \ge p \ge 0$.

Proof. For a particular value p, the most time-consuming step to compute $f(G, p)$ and $f_T(G, p)$ is when G is formed from G_1 and G_2 by a true twin operation. In the worst case, it takes at most $O(n^2)$ time to obtain the results. Therefore, it takes at most $O(n^3)$ time to compute $f(G, p)$ and $f_T(G, p)$ for all p, $n \ge p \ge 0$.

Q.E.D

Theorem 3.1 The edge domination problem on distance-hereditary graphs G can be solved in $O(n^4)$ time.

Proof. A one-vertex-extension tree of a distance-hereditary graph G can be computed in linear time. Once this has been done, we can obtain a recursive definition of the given graph in $O(n)$ time. Since a distance-hereditary graph can be obtained by performing $O(n)$ operations given in Definition 2.3, and it takes at most $O(n^3)$ time to compute $f(G, p)$ and $f_T(G, p)$ for $n \ge p \ge 0$ after each operation is performed. Consequently, the minimum edge dominating set $f(G, 0)$ of a distance-hereditary graph G can be obtained in $O(n^4)$ time.

4. REFERENCES

[1] E. Howorka, "A characterization of distance-hereditary graphs," Quart. J. Math. Oxford 28 (1977), 417-420.

[2] M. Yannakakis and F. Gavril, "Edge dominating sets in graphs", SIAM J. APPL. MATH., 38 (1980) 364-372.

[3] R. Laskar and K. Peters, "Vertex and edge domination parameters in graphs," Congressus Numerantium, 48 (1985) 291-305.

[4] H. J. Bandelt and H. M. Mulder, "Distance-hereditary graphs," J. Comb. Theory, Series B, 41 (1986) 182-208.

[5] M. B. Richey, R. G. Parker, "Minimum-maximal matching in series-parallel graphs," European Journal of Operational Research, 33 (1987) 98-105.

[6] P. L. Hammer and F. Maffray, "Comletely separable graphs," Disc. Appl. Math. , 27 (1990) 85-99.

[7] J. D. Horton and K. Kilakos, "Minimum edge dominating sets," SIAM J. DISC. MATH., 6 (1993) 375-387.

[8] D. L. Grinstead, P. J. Slater, N. A. Sherwani, and N. D. Holmes, "Efficient edge domination problems in graphs," Information Processing Letters, 48 (1993) 221-228.

[9] P. Dolan and M. Halsey, "Random edge domination," Discrete Mathematics, 112 (1993) 259-260.

[10] J. Topp, "Graphs with unique minimum edge dominating sets and graphs with unique maximum independent sets of vertices," Discrete Mathematics, 121 (1993) 199-210.

[11] Bor-Liang Chen, Hung-Lin Fu, "Edge domination in complete partite graphs," Discrete Mathematics, 132 (1994) 29-35.

[12] A. Srinivasan, K. Madhukar, P. Nagavamsi, C. Pandu Rangan, Maw-Shang Chang, "Edge domination on bipartite permutation graphs and cotriangulated graphs," Information Processing Letters, 56 (1995) 165-171.

[13] Maw-Shang Chang and Sun-Yuan Hsieh and Gen-Huey Chen, "Dynamic programming on distance-Hereditary graphs," Lecture Notes in Computer Science, 1350 (1997) 344-353.