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CIRCULAR TRAPEZOID GRAPHS AND CIRCLE TRAPEZOID GRAPHS: CHARACTERISTICS AND ALGORITHMS

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Abstract

Along with the direction that generalizes interval graphs and permutation graphs to (subclasses of) trapezoid graphs, researchers are now trying to generalize the class of trapezoid graphs. A *circle trapezoid* is the region in a circle that lies between two non-crossing chords, and the *circle trapezoid graphs* are the intersection graphs of circle trapezoids in a circle; note that circle trapezoid graphs properly contains trapezoid graphs, circle graphs and circular-arc graphs as subclasses. Circle trapezoid graphs shall not be confused with *circular trapezoid graphs*. Here a *circular trapezoid* is the region within *two* parallel circles that lies between two non-crossing segments . It follows that the circular trapezoid graphs are the intersection graphs of circular trapezoids between two parallel circles.

In this paper, the author presents results on two superclasses of trapezoid graphs including circle trapezoid graphs and circular trapezoid graphs. We show that circle trapezoid graphs and circular trapezoid graphs are two distinct classes of graphs; further, we show that the maximum weighted independent set on circular trapezoid graphs can be found in $O(n^2 \log\log n)$ time, and the minimum weighted independent dominating set of circular trapezoid graphs can be found in $O(n^2 \log n)$ time.

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1 Introduction

The intersection graph of a collection of trapezoids with corner points lying on two parallel lines is called the *trapezoid graph* [4, 6]. Note that trapezoid graphs are perfect and properly contain both interval graphs and permutation graphs. Trapezoid graphs are not necessary chordal since C_4 is a trapezoid graph; however, they are weakly chordal [5]. Recall that a graph G is weakly chordal if neither G nor $\overline{G}$ contains a chordless cycle of length ≥ 5. Dagan, Golumbic, and Pinter [6] show that the channel routing problem is equivalent to the coloring problems on trapezoid graphs and present an $O(n\chi)$ algorithm to solve it where χ is the chromatic number of the trapezoid graph.

The fastest known algorithm for recognition of trapezoid graph is given by Ma and Spinrad in [17], where they show that interval dimension 2 problem and trapezoid graphs recognition both can be solved in $O(n^2)$ time. That is, we can take the complement of the input graph, G, and use the transitive orientation technique (in $O(n^2)$ time) [18] to obtain a poset P and then tests whether P has interval dimension 2 in another $O(n^2)$ time. Since the given graph might not be a cocomparability graph, to avoid verifying the transitivity of $\overline{G}$, which takes $O(\mu(n))$ time, their algorithm needs to check the representation model in, again, $O(n^2)$ time. Habib and Möhring [10] also give an $O(n^3)$ time algorithm to recognize a trapezoid graph based on the $2-d$ interval order. Independently, using the vertex splitting technique, Cheah [2] also developed an $O(n^3)$ time algorithm for recognizing trapezoid graphs by graph theoretical approach.

Trapezoid graphs are perfect since they are cocomparability graphs. Thus the optimization problems including maximum independent set, clique, clique cover, and chromatic number of trapezoid graphs can all be solved in polynomial time by the ellipsoid method for perfect graphs [9]. Based on the geometric representation of trapezoid graphs by boxes in the plane, Felsner *et al.* [7] design $O(n \log n)$ time algorithms for chromatic number, weighted independent set, clique cover and maximum weighted clique for trapezoid graphs; the time can be improved to $O(n \log \log n)$ if the representations are sorted. It shall be noted that these results are also independently found by Chang [1]. Chen and Wang [3] show an algorithm for finding depth-first spanning trees on trapezoid graphs in $O(n)$ time. For the dominating sets problem and its variants in trapezoid graphs, see [13, 12, 19],

Along with the direction that generalizes interval graphs and permutation graphs to (subclasses of) trapezoid graphs, researchers are now trying to generalize the class of trapezoid graphs. For example, Flotow [8] introduces the class of m-trapezoid graphs that are the intersection graphs of m-trapezoids, where an m-trapezoid is given by $m+1$ intervals on $m+1$ parallel lines. Recall that the k-th *power* of a graph $G = (V, E)$, denoted G^k, is the graph with the same vertex while two vertices are adjacent iff there exists a path of length at most k connecting them. Flotow shows that if G^k is an m-trapezoid graph then G^{k+1} is also an m-trapezoid graph. Lin [14] show that determining whether a given graph is a k-th power graph for any fixed $k > 1$ is NP-complete.

Felsner *et al.* [7] generalizes their algorithms to m-trapezoid graphs (where they called it k-trapezoid graphs,) and give $O(n \log^{k-1} n)$ time algorithms for chromatic number, weighted independent set, clique cover and maximum weighted clique for k-trapezoid graphs. They also propose a new class of graphs called *circle trapezoid graphs,* also known as *circular strips graphs,* that properly contains trapezoid graphs, circle graphs and circular-arc graphs as subclasses; they propose an $O(n^2)$ time algorithm for weighted independent set and an $O(n^2 \log n)$ time algorithm for weighted clique problem for circle trapezoid graphs, using their algorithms for trapezoid graphs as subroutines. Note that a *circle trapezoid* is the region in a circle that lies between two non-crossing chords, and the circle trapezoid graphs are the intersection graphs of circle trapezoids in a circle. Just like *circular permutation graphs* [16] shall not be confused with circle graphs, circle trapezoid graphs shall not be confused with *circular trapezoid graphs,* defined by Kratsch [11]. Here a *circular trapezoid* is the region in *two* circles (parallel to each other, in the 3D space) that lies between two non-crossing segments (on the cylinder surface, connecting two endpoints in each circle.)

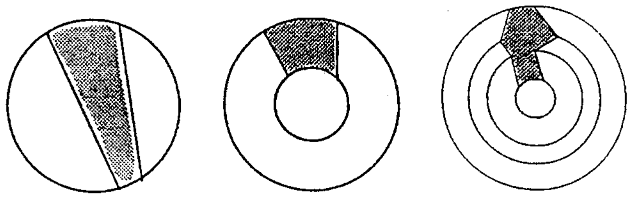

Figure 1: Circle trapezoid graphs, circular trapezoid graphs, and circular d-trapezoid graphs.

It follows that the circular trapezoid graphs are the intersection graphs of circular trapezoids between two parallel circles. They also extends circular trapezoid graphs into $d > 2$ parallel circles; the generalized classes of graphs is so called circular d-trapezoid graphs. Kratsch show that polynomial time algorithms for computing the *component number vectors* and the *maximum component order vectors* for measuring the 'vulnerability' of these graphs.

In summary, circular d-trapezoid graphs generalizes d-trapezoid graphs, but circular d-trapezoid graphs do not generalize circle trapezoid graphs. Note that d-trapezoid graphs are still cocomparability graphs, but circular d-trapezoid graphs and circle trapezoid graphs are not subclasses of cocomparability graphs. Further, it is still not known whether we can efficiently recognize circle trapezoid graphs, $(d > 2)$-trapezoid graphs, or circular $(d \geq 2)$-trapezoid graphs. It seems that research has been directed towards using the the specific topological or geometric structure of theses generalized trapezoid graphs to solve more intractable optimization problems in larger classes of graphs. Further, finding recognition algorithms on these variants of generalized trapezoid graphs will still be a challenge to the researchers. Some of the problems may have been partly answered [7, 11]; however, there may still be room for improvement, e.g., the weighted independent set and weighted clique problem for circle

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trapezoid graphs.

Most importantly, many optimization problems that can be efficiently solved in trapezoid graphs [3, 7, 12, 19] are still quite open to the researchers. Especially, little is known about how to efficiently solve any optimization problem for circular ($d \geq 2$)-trapezoid graphs, and not much is known about problems on circle trapezoid graphs.

In this paper, the author presents results on two superclasses of trapezoid graphs including circle trapezoid graphs and circular trapezoid graphs. The paper is organized as follows. In Section 2, we show that circle trapezoid graphs and circular trapezoid graphs are two distinct classes of graphs; actually, we discover that circular trapezoid graphs do *not* generalize circle trapezoid graphs. The second part of this paper concerns the algorithmic aspects of circular trapezoid graphs. In Section 3, we show that the maximum weighted independent set on circular trapezoid graphs can be found in $O(n^2 \log\log n)$ time. We show in Section 4 that the minimum weighted independent dominating set of circular trapezoid graphs can be found in $O(n^2 \log n)$ time.

2 Circle Trapezoid Graphs and Circular Trapezoid Graphs

To show that circle trapezoid graphs and circular trapezoid graphs are two distinct superclasses of trapezoid graphs, we first show that there are circle graphs that are not circular trapezoid graphs. In particular, we will show

Theorem 2.1 *The graph G shown in Figure 2 is a circle trapezoid graph (actually a circle graph), but it is* not *a circular trapezoid graph.*

Proof. From the model of G shown in Figure 2, it is easily seen that G is a circle graph (thus a circle trapezoid graph). Now we show that G is not a circular trapezoid graph.

Suppose that G is a circular trapezoid graph. Note that the outer six vertices induced a chordless simple cycle of length six (C_6) in G. It is not hard to verify that these corresponding six circular trapezoids in the circular trapezoid model must connect to each other in a fashion as the right hand side figure in Figure 2; i.e., these circular trapezoids will from a circu-

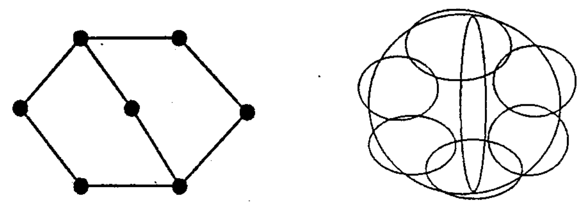

Figure 2: A circle trapezoid graph and its circle trapezoid model that is *not* a circular trapezoid graph.

lar chain of length six in the circular channel. Now consider the seventh vertex of G (the middle vertex) shown in Figure 2. The corresponding circular trapezoid must intersect two opposite circular trapezoids of the circular chain but not intersecting any of the middle four circular trapezoids. However, this can not be done because these outer six circular trapezoids form a continuous circular chain with two opposite circular trapezoids not intersecting each other; thus, any continuous curve intersecting two opposite circular trapezoids must also (at least) intersecting two other middle circular trapezoids. Actually, it is not hard to generalize the result to a family of graphs that are circle (trapezoid) graphs but not circular trapezoid graphs. □

We use the notation $u \sim_G v$ to represent u and v are two adjacent vertices in G, i.e., $\{u, v\} \in E(G)$. When the underlying G is clear, we will drop the subscribe, and just write $u \sim v$. Further, given two subset of vertices A, B, we generalize the notation in $A \sim B$ to mean that $a \sim b$ for all vertices $a \in A$ and $b \in B$. To show that there are circular trapezoid graphs that are not circle trapezoid graphs, and thus showing that circle trapezoid graphs is distinct from circular trapezoid graphs, we need the following property of circle trapezoid graphs (which also applies to circular trapezoid graphs as well):

Lemma 2.2 (X-shape) *Let $v_1, \ldots, v_6$ be six distinct vertices of a circle (circular) trapezoid graph G such that their induced subgraph form a $K_{3,3}$ with $\{v_1, v_2, v_3\} \sim \{v_4, v_5, v_6\}$. Let S (T) be the middle trapezoid of the three trapezoids corresponding to vertices $\{v_1, v_2, v_3\}$ ($\{v_4, v_5, v_6\}$) in the (circle, circular) trapezoid model of G. Then the upper (lower) interval of S is disjoint with the upper (lower) interval of T.*

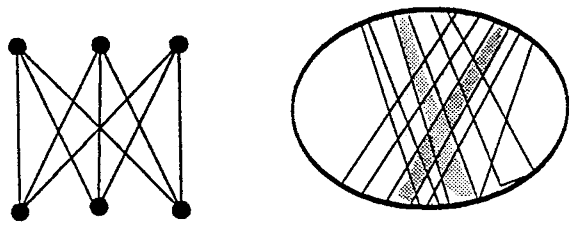

Figure 3: A $K_{3,3}$ induced subgraph in circle (circular) trapezoid graph shall force an "X"-shape of two middle circle (circular) trapezoids t_2 and t_5.

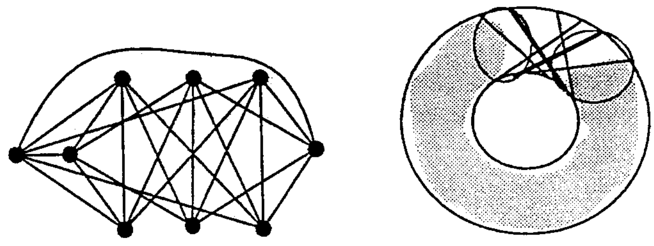

Figure 4: A circular trapezoid graph and its circular trapezoid model that is *not* a circle trapezoid graph.

Proof. Let t_i represent the trapezoid corresponding to the vertex v_i for $i \in [1..6]$ in the trapezoid model of G. Note that vertices v_1, v_2 and v_3 are independent vertices in $K_{3,3}$. The corresponding circle trapezoids can either be 3 arcs; i.e., there is no chord intersecting these 3 circle trapezoids at the same time. Or, these circle trapezoids shall be *parallel* i.e., there is one chord intersecting these 3 circle trapezoids at the same time, as shown in the right hand part of Figure 3. However, if these 3 circle trapezoids are 3 arc-trapezoids, then it will be impossible for the other three independent vertices, namely v_4, v_5, v_6, intersecting all vertices of v_1, v_2 and v_3. That is, we conclude that the 3 circle trapezoids t_1, t_2, t_3 (and thus t_4, t_5, t_6) are 3 parallel circle trapezoids.

Denote $t_i \parallel t_j$ if t_i does not intersect with t_j. Further, denote $t_i \parallel t_j \parallel t_k$ if t_i, t_j, t_k are 3 parallel circle trapezoids with t_j being the middle circle trapezoid. Without loss of generality, we assume that $t_1 \parallel t_2 \parallel t_3$ and $t_4 \parallel t_5 \parallel t_6$; note that $t_2 = S$ and $t_5 = T$.

Assume that t_2 does intersect t_5 in the lower (upper) interval as illustrated by the diagram shown in Figure 3. Since t_2, t_5 intersect each other on the lower interval, the lower interval of t_4 lies on the left to the lower interval of t_3. By the same reason, the lower interval of t_6 lies on the right to the lower interval of t_1.

However, since $v_1 \sim v_6$ and $v_3 \sim v_4$, it implies that t_1 and t_6, also t_3 and t_4, intersect each other on the upper intervals, which is impossible for $t_1 \parallel t_3$ and $t_4 \parallel t_6$.

By symmetry, we reach the same contradiction if we assume t_2 and t_5 intersect in the upper intervals. That is, both the upper and lower intervals of T_1 and T_2 are disjoint to each other. In other words, they intersect each other by the "X" shape. □

Using this "X"-Shape Lemma (or the $K_{3,3}$ Lemma) as a gadget, we are able to design a circular trapezoid graphs which do not have circle trapezoid representation. In particular,

Theorem 2.3 *The graph G shown in Figure 4 is a circular trapezoid graph, but it is* not *a circle trapezoid graph.*

Proof. From the model of G shown in Figure 4, it is easily seen that G is a circular trapezoid graph. Now we show that G is not a circle trapezoid graph.

Note that the middle six vertices of G, shown in Figure 4, induce a $K_{3,3}$. Denote the top vertices by a, b, c; note that vertices a, b, c have degree 5. Denote the bottom vertices by d, e, f; note that vertices d, e, f have degree 5. Denote the rest 3 vertices by x, y, z; note that vertices y, z have degree 5 and the vertex x has degree 6.

Suppose that G is a circle trapezoid graph. By the X-shape Lemma 2.2, a, b, c are three parallel, independent, circle trapezoids. Note that the vertex y is adjacent to both vertices a and b, but not vertex c. It follows that the corresponding circle trapezoid c can not be the middle circle trapezoid. The reason is that, if a trapezoid intersecting both end of the 3 parallel trapezoid, it will definitely intersecting the middle one as well. Further, since z is adjacent to both vertices b and c, but not vertex a. It follows that the corresponding circle trapezoid a can not be the middle circle trapezoid. Thus we conclude that $a \parallel b \parallel c$. By the same reasoning, we also have: $d \parallel e \parallel f$.

However, note that we have a vertex x intersect both vertices a and c, but not intersect vertex b, which leads us to the contradiction. Thus we conclude that G cannot be a circle trapezoid graph. □

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Combining with Theorem 2.1, we have:

Corollary 2.4 *Circle trapezoid graphs and circular trapezoid graphs are two distinct superclasses of trapezoid graphs.*

3 Independent Set of Circular Trapezoid Graphs

The second part of this paper concerns the algorithmic aspects of circular trapezoid graphs. In this section, we will show that the maximum weighted independent set on circular trapezoid graphs can be found in $O(n^2 \log\log n)$ time.

Assume that we are given a set of n circular trapezoids $T = \{t_1, \ldots, t_n\}$. The model of each circular trapezoid $t \in T$ is represented by five tuples (a, b, c, d, s), with $a, b, c, d \in \{1..2n\}, s \in \{+, -\}$; we will use $t.a, t.b, t.c, t.d, t.s$ to denote these five values. For the weighted version of the maximum independent set problem, each circular trapezoid t is associated with a (positive) real weight $w(t)$. Note that $(t.a, t.b)$ represents the *circular arc* (interval) of the *outer* circle in the circular trapezoid model, clockwise connecting the point $t.a$ to the point $t.b$; in the same notion, $(t.c, t.d)$ represents the circular arc of the *inner* circle. Note that given two circular arcs s, t, with one in the outer circle and the other in the inner circle, there are *two* different ways of connecting these two arcs into a circular trapezoid. Either we can connect s and t clockwise, or counterclockwise. Thus, we use $+$ or $-$ signs to represent the connecting ways accordingly.

Given a vertex $v \in V$, define the neighbors of v as $N(v) = \{u : (u, v) \in E\}$; the closed neighbors of v is defined by $N[v] = N(v) \cup \{v\}$. Assume that we are given a subset $I \subset V$ that defines an independent set in the underlying circle trapezoid model. It is easily verify that:

Observation 3.1 *Given a graph $G = (V, E)$, let subset $I \subset V$ be the maximum independent set of G with a vertex $v \in I$. Let H be the subgraph of G induced by vertices $V \setminus N[v]$. It follows that $I \setminus \{v\}$ is the maximum independent set of H.*

Proof. By contradiction. Assume that there were a larger weighted independent set I' in subgraph H. Clearly, $I' \cup \{v\}$ will be a larger weighted independent set in G, which is impossible. □

Given a circle trapezoid v of the circle trapezoid model, the subgraph, H, induced by vertices $V \setminus N[v]$ will be just a normal trapezoid graph. Note that we can find the maximum weighted independent set of H in $O(n \log\log n)$ time [7]. It follows that we can iterate through all possible candidate vertex of v; and find the maximum weighted independent set of circular trapezoid graphs in $O(n^2 \log\log n)$ time. The algorithm is shown in Figure 5. It follows that

Theorem 3.2 *Finding the maximum weighted independent set in a circular trapezoid graph can be done in $O(n^2 \log\log n)$ time and $O(n)$ space.*

4 Independent Dominating Set of Circular Trapezoid Graphs

A *dominating set* of a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one vertex in D. Each vertex $v \in V$ can be associated with a (non negative) real weight, denoted by $w(v)$. The *weighted domination problem* is to find a dominating set, D, such that its weight $w(D) = \sum_{v \in D} w(v)$ is minimized. An independent dominating set D is a dominating set that no two vertices of D are adjacent in G. That is, an independent dominating set is a dominating set as well as an independent set in G.

For finding the minimum independent dominating set in circular trapezoid graphs, we use the same idea as we have developed in Section 3. That is, given a circle trapezoid v of the circle trapezoid model, the subgraph, H, induced by vertices $V \setminus N[v]$ will be just a normal trapezoid graph. Note that we can find the minimum weighted independent dominating set of trapezoid graph in $O(n \log n)$ time [15]. It follows that we can iterate through all possible candidate vertex of v, and find the minimum weighted independent dominating set of circular trapezoid graphs in $O(n^2 \log n)$ time. The algorithm is shown in Figure 6.

Theorem 4.1 *Finding the minimum weighted independent dominating set in a circular trapezoid graph can be done in $O(n^2 \log n)$ time and $O(n)$ space.*

ALGORITHM **WIS(T)**

Input: A set of n circular trapezoids $T = \{t_1, \ldots, t_n\}$. Each circular trapezoid t_i is represented by five tuples (a, b, c, d, s), with $a, b, c, d \in \{1..2n\}, s \in \{+, -\}$. Each circular trapezoid t is associated with a (positive) real weight $w(t)$.

Output: A subset $I \subset T$ such that I is the maximum weighted independent set in the intersecting circular trapezoid graphs defined by T.

Step 1: For each circle trapezoid $v \in T$, remove every circle trapezoids of $N[v]$ from the circle trapezoid model. The resulting graph will be a trapezoid graph H.

Step 2: Find the maximum weighted independent set, $IS(H)$, in the trapezoid graph H. The proposed independent set has a weight: $W(v) = w(v) + \sum_{u \in IS(H)} w(u)$.

Step 3: Among all circle trapezoids, find the vertex, v, with the largest extended weight $W(v)$. It follows that $\{v\} \cup IS(H)$ is the maximum weighted independent set.

END OF **WIS**

Figure 5: Finding the maximum weighted independent set in a circular trapezoid graph.

ALGORITHM **IDS(T)**

Input: A set of n circular trapezoids $T = \{t_1, \ldots, t_n\}$. Each circular trapezoid t_i is represented by five tuples (a, b, c, d, s), with $a, b, c, d \in \{1..2n\}, s \in \{+, -\}$. Each circular trapezoid t is associated with a (positive) real weight $w(t)$.

Output: The minimum weighted independent dominating set of the circular trapezoid graph.

Step 1: For each circle trapezoid $v \in T$, remove every circle trapezoids of $N[v]$ from the circle trapezoid model. The resulting graph will be a trapezoid graph H.

Step 2: Find the minimum weighted independent dominating set, $ID(H)$, in the trapezoid graph H. The proposed independent set has a weight: $W(v) = w(v) + \sum_{u \in ID(H)} w(u)$.

Step 3: Among all circle trapezoids, find the vertex, v, with the smallest $W(v)$. It follows that $\{v\} \cup ID(H)$ is the minimum weighted independent dominating set.

END OF **IDS**

Figure 6: Finding the minimum weighted independent dominating set in a circular trapezoid graph.

5 Concluding Remarks

In this paper, we show that circle trapezoid graphs and circular trapezoid graphs are two distinct classes of graphs; actually, we discover that circular trapezoid graphs do *not* generalize circle trapezoid graphs. Further, we show that the maximum weighted independent set on circular trapezoid graphs can be found in $O(n^2 \log\log n)$ time, and the minimum weighted independent dominating set of circular trapezoid graphs can be found in $O(n^2 \log n)$ time.

As we have discussed in Section 1, many optimization problems that can be efficiently solved in trapezoid graphs [3, 7, 12, 19] are still quite open to the researchers as whether these problems can be solved efficiently in the generalized trapezoid graphs. Especially, finding recognition algorithms on these variants of generalized trapezoid graphs can still be a challenge to the researchers.

References

[1] Maw-Shang Chang. Efficient algorithms for trapezoid graphs. Manuscript, 1994.

[2] F. Cheah and D.G. Corneil. On the structure of trapezoid graphs. *Discr. Applied Math.*, 66:109–133, 1996.

[3] Hon-Chan Chen and Yue-Li Wang. A linear time algorithm for finding depth-first spanning trees on trapezoid graphs. *Information Processing Letters*, 63(1):13–18, 14 July 1997.

[4] D.G. Corneil and P.A. Kamula. Extensions of permutation and interval graphs. In *Proc. 18th Southeastern Conference on Combinatorics, Graph theory and Computing*, pages 267–276, 1987.

[5] D.G. Corneil, S. Olariu, and L. Stewart. On the linear structure of graphs. In *The 1st Workshop on COST*, pages 99–106. DIMACS/RUTCOR, April 1991.

[6] I. Dagan, M.C. Golumbic, and R.Y. Pinter. Trapezoid graphs and their coloring. *Discr. Applied Math.*, 21:35–46, 1988.

[7] Felsner, Muller, and Wernisch. Trapezoid graphs and generalizations, geometry and algorithms. *Discr. Applied Math.*, 74, 1997.

[8] Flotow. On powers of m-trapezoid graphs. *Discr. Applied Math.*, 63, 1995.

[9] M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1:169–197, 1981.

[10] M. Habib and R.H. Möhring. A fast algorithm for recognizing trapezoid graphs and partial orders of interval dimension 2. Preprint, 1990.

[11] Dieter Kratsch, Ton Kloks, and Haiko Müller. Measuring the vulnerability for classes of intersection graphs. *Discr. Applied Math.*, 77:259–270, 1997.

[12] Y. D. Liang. Steiner set and connected domination in trapezoid graphs. *Information Processing Letters*, 56(2):101–??, 1995.

[13] Y. Daniel Liang. Dominations in trapezoid graphs. *Information Processing Letters*, 52(6):309–315, December 1994.

[14] Y.-L. Lin. Recognizing powers of graphs is hard. In *The 12th Workshop on Combinatorial Mahtematics and Computation Theory*, pages 120–127, Kaohsiung, Taiwan, 1995.

[15] Yaw-Ling Lin. Fast algorithms for independent domination and efficient domination in trapezoid graphs. Manuscript, 1998.

[16] Lou and Sarrafzadeh. Circular permutation graph family with applications. *DAMATH: Discrete Applied Mathematics and Combinatorial*

Operations Research and Computer Science, 40, 1992.

[17] T.-H. Ma and J.P. Spinrad. An $O(n^2)$ time algorithm for the 2-chain cover problem and related problems. In *Proc. 2nd ACM-SIAM Symp. Discrete Algorithms*, pages 363–372, 1991.

[18] J. Spinrad. On comparability and permutation graphs. *SIAM Journal on Computing*, 14:658–670, 1985.

[19] Anand Srinivasan, M.S. Chang, K. Madhukar, and C. Pandu Rangan. Efficient algorithms for the weighted domination problems on trapezoid graphs. Manuscript, 1996.