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An Efficient Algorithm for the Connected Two-Center Problem*

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Abstract

This paper considers the connected two-center problem, which is to find two congruent closed discs of smallest radius whose union covers a set of n given points in the plane and whose centers are close as specified connectivity. The previously best known algorithm for this problem is straightforward and based on the exhaustive searching paradigm which leads to $O(n^5)$ time complexity. In this paper, we design an $O(n^3)$ time algorithm for solving the problem by using a new data structure called *center-hull*, which has rather close relationships with the farthest-point Voronoi diagram. The properties of center-hull are also discussed in this paper.

Keywords: Computational Geometry, Connected Two-Center problems, Center Hulls, Farthest-Point Voronoi Diagram

1 Introduction

Let N be a set of n points in the plane. The k-center problem is to cover N by k congruent closed discs whose radius is as small as possible. The problem was proved to be NP-complete when the parameter k is a part of the input [19]. A recent best known algorithm for this problem is given in [15] with time complexity $O(n^{o(\sqrt{k})})$. The two-center problem is a special case of the general k-center problem and is much simpler. However, it is a long time for authors to solve this problem [4, 1, 8, 16, 17, 22, 9, 11, 3]. In 1996, Sharir [22] proposed a near-linear time algorithm for the two-center problem, and in 1997, Eppstein [9] proposed a randomized algorithm with $O(n \log^2 n)$ expected time for the problem. The important applications of the two-center problem include transportation, station placement and facility location [4, 16, 3] .

In this paper, we consider a restricted version of the two-center problem, namely, connected two-center problem (abbreviate to C2C problem). Given a set N of n points in the plane, the C2C problem is to find two congruent closed discs of smallest radius whose union covers N and whose centers are close as specified connectivity. One of applications of the C2C problem is to set up the medical centers. Suppose that we want to place two medical emergency units so that the worst-case response time to each of n given sites is minimized. If the two units are well-independent, then everything will be okay. However, in realistic environment, it is possible that one of the unit may request blood (or technical support) from the other unit for emergency surgical operations. As a result, to shorten the distance of the two units seems somewhat necessary.

Huang [13] is the first researcher to discuss this problem, and he gave an algorithm with $O(n^5)$ time complexity for this problem. We shall present a new algorithm for solving the C2C problem which runs in $O(n^3)$ time. Our new algorithm is mainly based on the farthest-point Voronoi diagram [20] and a new data structure called center-hull, which is similar to circular-hull [9, 12, 7, 6]. The new algorithm is conceptually much simpler and has more explicit geometric flavor.

This paper is organized as follows. In Section 2 we introduce the constrained C2C problem. For solving the constrained C2C problem, we proposed a new data structure, center-hull, in Section 3. In addition, the properties of center-hull are also exploited. The algorithm for solving the constrained C2C problem is given in Section 4. Some open problems and directions for further research will be presented in Section 5.

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2 The constrained C2C problem

In this section, we shall describe how we reduce the C2C problem into the *constrained C2C problem* (abbreviate to CC2C problem). We solve the C2C problem by firstly dividing the problem into $O(n^2)$ instances of CC2C problem. Then we aim to develop an algorithm for solving the CC2C problem and apply it $O(n^2)$ times.

To proceed further we need some definitions that are necessary to formulate the CC2C problem and that are also used in the description of the rest of this paper.

Definition 1 The point set $N = \{p_1, p_2, \ldots, p_n\}$ is a set of n points in the plane.

Definition 2 $C(p, r)$ denotes the circle centered at point p with radius r.

Definition 3 Circle $C(p, r)$ is called to *cover* N if $|pq| \leq r$ for each $q \in N$.

Definition 4 The *connected two-circle* C2C(x, y, r) is a pair of circles, $C(x, r)$ and $C(y, r)$, such that $0 \leq |xy| \leq 2r$. Note that C2C(x, y, r) becomes the one-center problem if $|xy|$ equals to 0.

Definition 5 The C2C(x, y, r) is called to *cover* N if $|px| \leq r$ or $|py| \leq r$ for each $p \in N$.

Definition 6 The *connected ratio* α of the C2C(x, y, r) is defined to be $1 - |xy|/2r$. And we further call the two circles *α-connected two-circle*, denoted as α-C2C(x, y, r).

Definition 7 The α-C2C(x, y, r) is said to be *strongly connected* if $1/2 \leq \alpha \leq 1$; and for $0 \leq \alpha < 1/2$, it is said to be *weakly connected*.

In Figure 1, we show three examples of α-connected two-circle α-C2C(x, y, r). ¿From above definitions, we can redefine the C2C problem as follows:

Definition 8 *Given a set N of n points in the plane and the value of α, the connected two-center problem is to find an α-connected two-circle α-C2C(x, y, r) covering N such that r is minimized.*

Definition 9 The *linear partition* (N_1, N_2) of a point set N is partitioning N into two subsets, N_1 and N_2, with a straight line such that $N_1 \cap N_2 = \phi$ and $N_1 \cup N_2 = N$.

Definition 10 Given a linear partition (N_1, N_2) of N and the value of α, the constrained connected two-center (CC2C) problem is to find an α-C2C(x, y, r) such that N_1 and N_2 can be covered by $C(x, r)$ and $C(y, r)$, respectively, and r is minimized.

For simplicity, from now on we omit specifying the α factor in the following description, and this does not affect the correctness of our discussions. The reader could think our description to be for the 1/2-connected case. However, it is, in fact, also applicable for all the cases of α.

Theorem 1 Given a point set N, there exists a CC2C problem whose solution is same as the C2C problem.

Proof. We prove it by generating a CC2C problem which has the same solution as the orginal C2C problem. Let C2C(x, y, r) be the solution of the C2C problem, we can generate a linear partition (N_1, N_2) by drawing a straight line l which is the perpendicular bisector of the line segment xy, that is, line l will pass through the cross points of the two circles $C(x, r)$ and $C(y, r)$. Obviously, N_1 and N_2 can be covered by $C(x, r)$ and $C(y, r)$ respectively. The solution radius r^* of the CC2C problem with linear partition (N_1, N_2) can not be less than r, otherwise C2C(x, y, r^*) will become the better solution to the orginal C2C problem. Hence, C2C(x, y, r) is the solution of the CC2C problem with linear partition (N_1, N_2). Q.E.D.

The immediate consequence of the above theorem is that we can solve the C2C problem of a given point set by solving all of the possible instances of CC2C problem, each with different linear partition. The number of possible partitions is asymptotically $O(n^2)$. In the following sections, we shall first introduce the notion of center-hull. Then we show that the CC2C problem is, in fact, a distance problem between two center-hulls.

3 The center-hull

The α-hull(also known as circular-hull) [9, 12, 7, 6] of a set N of n points in the plane is the intersection of all closed discs with radius $1/\alpha$ that contain all the points of N, where $\alpha > 0$. In this section, we consider a variant of this concept. A new data structure, center-hull, is proposed. We shall firstly introduce the notion and properties of center-hull. Then we introduce the close relationship between center-hull and farthest-point Voronoi diagram (FPVD) [20], which is crucial to the development of our algorithm.

Definition 11 The *center-hull* of N with radius r, denoted $CH(N,r)$, is the locus of points which are the centers of the circles with radius r that cover N, that is, $CH(N,r) = \{p|C(p,r) \text{ covers } N\}$.

The following two propositions are quite straightforward.

Lemma 1 Let r_0 be the radius of the smallest enclosing circle of N. The $CH(N,r)$ is empty if $r < r_0$.

Lemma 2 The point p is in $CH(N,r)$ if and only if the distance between p and its farthest neighbor in N is less than or equal to r.

There are also several properties of center-hull that will be useful, We state them below.

Theorem 2 The center-hull is convex.

Proof. Let a, b be any two points in $CH(N,r)$ and p be a point on ab. Furthermore, let the farthest neighbors in N of a, b and p be A, B and P respectively, and thus we have $|aP| \le |aA|$ and $|bP| \le |bB|$. Obviously, $|pP| < \max\{|aP|, |bP|\}$. Then by Lemma 2 and from the above inequalities, we have $|pP| < \max\{|aA|, |bB|\} \le r$. So the point p is also in $CH(N,r)$, this completes the proof. Q.E.D.

Lemma 3 A point p is on $\partial CH(N,r)$, the boundary of the $CH(N,r)$, if and only if the distance between p and its farthest neighbor in N is equal to r.

Proof. It follows immediately from Lemma 2. Q.E.D.

The following description of the relationships between center-hull and FPVD of a given point set stems from Lemma 3.

Definition 12 Let p be a point in N. The farthest-neighbor Voronoi region associated to p is the open region in the FPVD of N so that p is the farthest neighbor in N of all the points in the region.

Lemma 4 The $\partial CH(N,r)$ within each farthest-neighbor Voronoi region associated to p is an subarc centered at p with radius r.

Proof. It follows from the properties of FPVD and Lemma 3. Q.E.D.

Theorem 3 The $\partial CH(N,r)$ is a circular list of subarcs each centered at the associated farthest-neighbor in N with radius r.

Proof. Each farthest-neighbor Voronoi region of FPVD is associated to one vertex of the convex hull [21], that is, each vertex of the convex hull is the farthest neighbor in N to the associated farthest-neighbor Voronoi region. Thus the farthest-neighbor Voronoi regions can be ordered as a circular list of the vertices of the convex hull. Then by Lemma 4, $\partial CH(N,r)$ can be found by taking a round trip over the circular list of the farthest-neighbor Voronoi regions and the vertices of the convex hull. Q.E.D.

In Figure 2, we show an example of the FPVD together with the $CH(N,r)$ for $N = \{a,b,c,d\}$.

Let the center of each subarc on $\partial CH(N,r)$ be called the *control point* of that subarc. Then the $CH(N,r)$ can be described by the circular list of the control points together with the radius r. We refer to this circular list of the control points as the *control point configuration* $CP(N,r)$. For the same FPVD, the control point configuration differs from one to others, due to the changes of the value of r.

Let r_0 be the radius of the smallest enclosing circle of N. The number of points in the control point configuration may increases as the value of r grows. The value of r at which the control point configuration changes is called a *breaking radius*. A breaking radius occurs while the fatting center-hull intersects with a new vertex of the FPVD, so we can find all of the breaking radii by tracing along the edges of the FPVD. The number of vertices of the FPVD is asymptotically $O(n)$, so the number of breaking radii is the same. By sorting on the values of breaking radii, we obtain a *breaking radius sequence* of N.

4 Solving the CC2C problem

In fact, the center-hull $CH(N,r)$ is indeed the locus of centers at which the circles covering N centered (with radius r). In this section, we present an algorithm for solving the CC2C problem. Key to our approach is to transform the CC2C problem into a distance problem between two center-hulls.

Definition 13 The distance between $CH(N_1,r)$ and $CH(N_2,r)$ is $min_{p,q}|pq|$ where $p \in CH(N_1,r)$ and $q \in CH(N_2,r)$, and p, q are called the nearest points of the two center-hulls.

From above definition, it is easy to see that the nearest points of two center-hulls lie on the boundaries of the center-hulls.

Theorem 4 If r^* is the minimal r such that the distance between $CH(N_1, r)$ and $CH(N_2, r)$ is equal to $2r(1-\alpha)$, then there exists a α-connected two-circle α-C2C(p, q, r^*) that is a solution to the α-connected CC2C problem with linear partition (N_1, N_2).

Proof. Let p, q be the nearest points of $CH(N_1, r^*)$ and $CH(N_2, r^*)$ where p in $CH(N_1, r^*)$ and q in $CH(N_2, r^*)$. ¿From the property of center-hull as described at the beginning of this section, we can draw two circles $C(p, r^*)$, $C(q, r^*)$ that cover N_1 and N_2 respectively. Moreover, since $|pq| = 2r^*(1-\alpha)$ and r^* is minimized, $C(p, r^*)$ and $C(q, r^*)$ form a α-connected two-circle α-C2C(p, q, r^*). Then by the definition of constrained connected two-center problem (Definition 10), it follows. Q.E.D.

Theorem 4 gives us the idea to solve the CC2C problem (with given linear partition) by solving the distance problem between the two corresponding center-hulls.

Let c_1 in $CH(N_1, r)$ and c_2 in $CH(N_2, r)$ be the nearest points of the two center-hulls. Before proceeding, we further define the *dual critical point set* DCPS(N_1, N_2, r) as (S_1, S_2), where S_i is the set of the farthest neighbors of c_i in N_i for $i = 1, 2$.

Since each nearest point of the two center-hulls lies either on an edge (subarc) or on a vertex (the intersection of two subarcs), we have the following corollary.

Corollary 1 There are one or two points in S_i of the DCPS of two center-hulls for $i = 1, 2$.

Given a linear partition (N_1, N_2) of a point set N and the value of r, $CH(N_1, r)$ and $CH(N_2, r)$ are defined. Since the center-hull has the convex property as the convex n-gons, we can use the algorithm given in [2] to find the nearest points of two center-hulls (and so DCPS) by replacing the straight lines of the convex n-gons with the subarcs of the center-hulls. Inversely, suppose DCPS$(N_1, N_2, r^*) = (S_1^*, S_2^*)$. If S_1^* and S_2^* are known, we can find the value of r^* by solving a polynomial equation with degree no more than four. For more detail about solving such polynominals, see [5].

Here we start to describe our algorithm. For a given linear partition (N_1, N_2) of a point set N, the constrained connected two-center algorithm consists of the following three phases.

Let r_1 and r_2 be the smallest enclosing radius for N_1 and N_2 respectively, and without loss of generality, assume that $r_1 \geq r_2$. The first phase checks whether r_1 is the solution, i.e., there exists two circles C_i with radius r_1 that covers N_i respectively, for $i = 1, 2$, and they are also α-connected as demanded. If it does, we are done and $r^* = r_1$ is the solution radius, otherwise proceed to the next phase.

The second phase finds the range (r_l, r_h) that r^* lies in and both $CP(N_1, r)$ and $CP(N_2, r)$ keep unchanged for r in (r_l, r_h). This phase is composed of many steps which will be outlined later.

The third phase is to find the exact r^* in the range (r_l, r_h). After phase 2, we have only ensured that both shapes of the two center-hulls keep unchanged for r in the range (r_l, r_h). Since the DCPS(N_1, N_2, r_l) may not equal to the DCPS(N_1, N_2, r^*), so we cannot compute r^* directly from DCPS(N_1, N_2, r_l). Figure 3 shows an example that DCPS changes from $(\{A\}, \{B, C\})$ into $(\{A\}, \{B\})$ while the radius grows from r to r', where p, q (respect to p', q') are the nearest points of the two center-hulls with radius r (respect to r'). Let's now disscuss on how we find the DCPS(N_1, N_2, r^*) and the value of r^*.

In the third phase, we gradually increase the radius value starting from r_l. Let r be the current radius value, and the current dual critical point set be DCPS(N_1, N_2, r). By taking a look on $CH(N_1, r+\varepsilon)$ and $CH(N_2, r+\varepsilon)$ where ε is a very small value, we can find out the change of relative positions of the nearest points of the two center-hulls. This change gives us the information that how the dual critical point set would change while r is growing. Then by computing on the boundary conditions, we can find the next radius value r' $(r' > r)$ which makes the corresponding dual critical point set to be changed into the other configuration, denoted as DCPS(N_1, N_2, r'). This process is repeated until the DCPS(N_1, N_2, r^*) is reached. Then we can compute r^* directly from DCPS(N_1, N_2, r^*) as described above.

Definition 14 An element of center-hull is a vertex or an edge of the center-hull.

Definition 15 The distance between two elements of center-hulls is the distance between the nearest points of the two elements.

Definition 16 Let a_i be an element of $CH(N_i, r)$ for $i = 1, 2$. Let a_i' be an element of $CH(N_i, r+\varepsilon)$ and has the same control point with a_i, where $\varepsilon > 0$, such that $|a_i a_i'| = \varepsilon$ for $i = 1, 2$. The *potential* of a_1 and a_2, denoted by $P(a_1, a_2)$, is defined as $|a_1 a_2| - |a_1' a_2'|$.

Figure 4 shows the case that the two center-hulls are just two circles. It is easy to see that $P(a_{im}, b_{in}) \geq P(a_{ip}, b_{iq})$ if $m \leq p$ and $n \leq q$ for $i = 0, 1$. In particular, $P(a_{00}, b_{00}) = 2\varepsilon$ is the highest potential, and

$P(c,d) = 0$ implies that $|cd|$ would never be changed while r is growing.

Observation 1. For two center-hulls, $CH(N_1,r)$ and $CH(N_2,r)$, there exists two elements $a \in CH(N_1,r)$ and $b \in CH(N_2,r)$ such that a, b have the highest potential. And for elements $c \in CH(N_1,r)$ and $d \in CH(N_2,r)$, the $P(c,d)$ monotonically decreasing while c getting farther from a or d getting farther from b.

The changes of the DCPSs in the third phase correspond to the changes of the nearest points on the relative positions of the two center-hulls. Let p, q be the nearest points of the current two center-hulls, and p', q' be the next nearest points. Obviously, $P(p',q') > P(p,q)$, otherwise p', q' cannot become the new nearest points. It is true that the next nearest points always have higher potential than the current nearest points while r is growing. Then from observation 1, we conclude that the moving directions of the two nearest points are both monotonically toward the positions of highest potential. Thus, the repetitions of tracing the changes of DCPSs would be definitely bounded.

Lemma 5 *If $DCPS(N_1,N_2,r) = (S_1,S_2)$ and $||S_1|| = ||S_2|| = 1$, then $DCPS(N_1,N_2,r') = (S_1,S_2)$ for $r' > r$.*

Proof. The case $||S_1|| = ||S_2|| = 1$ implies that the two critical points and the two nearest points are colinear. And so the potential of the two nearest points equals to 2ε. Since 2ε is the maximum value for the potential, the two nearest points would stay nearest while r is growing. And hence the DCPS(N_1,N_2,r') would never be changed for $r' > r$. Q.E.D.

Now we outline the algorithm for solving the constrained connected two-center problem below.

Algorithm: The constrained connected two-center algorithm

Input: A point set N, the linear partition (N_1,N_2) and the connected ratio α.

Output: the minimal radius r^*

Phase 1:

Step 1.1 Let the radius of the smallest enclosing circle for N_i is r_i, for $i = 1,2$.

Step 1.2 Let $r = \max\{r_1,r_2\}$ and the distance between $CH(N_1,r)$ and $CH(N_2,r)$ be d.

Step 1.3 If $d \leq 2r(1-\alpha)$ then $r^* = r$ is the solution, otherwise proceed to Phase 2.

Phase 2:

Step 2.1 Construct the farthest-point Voronoi diagram for N_i , and then compute the breaking radius sequence R_i of N_i , for $i = 1,2$.

Step 2.2 Merge (sort) R_1 and R_2 into an ascending sequence $R = \{r_1,r_2,\ldots,r_m\}$.

Step 2.3 Let $min = 1$ and $max = m$, and let $mid = (min + max + 1)/2$.

Step 2.4 Let the distance between $CH(N_1,r_{mid})$ and $CH(N_2,r_{mid})$ be d.

Step 2.5 If $2r_{mid}(1-\alpha) < d$ then $min = mid$ else $max = mid$.

Step 2.6 Let $mid = (min + max + 1)/2$.

Step 2.7 If $max \neq mid$ then go to Step 2.4, else proceed to Phase 3.

Phase 3:

Step 3.1 Let $r = r_{min}$, p and q be the current nearest points of $CH(N_1,r)$ and $CH(N_2,r)$, and $dcps = DCPS(N_1,N_2,r)$.

Step 3.2 Compute the next dual critical point set $dcps'$, the next radius value r' and the new nearest points p' and q' as described above.

Step 3.3 If $|p'q'| > 2r'(1-\alpha)$ then let $r = r'$, $p = p'$, $q = q'$, $dcps = dcps'$ and go to Step 3.2, else proceed to Step 3.4.

Step 3.4 Compute r^* from $dcps$.

Theorem 5 The constrained connected two-center algorithm takes $O(n \log n)$ time.

Proof. Step 1.1 can be completed in $O(n)$ time by using the one-center algorithm posed in [18]. In step 1.2, the two center-hulls can be computed in $O(n)$ time, and their distance can be found in $O(\log n)$ time [2]. Phase 2 is dominated by the time to construct the two farthest-point Voronoi diagrams which takes $O(n \log n)$ time. There are at most $O(n)$ iterations in Phase 3 since the moving directions of the tracing are monotonic, and each iteration takes $O(1)$ time. Therefore, the time complexity of the constrained connected two-center algorithm is $O(n \log n)$ *Q.E.D.*

There are only $O(n^2)$ ways to partition an n-point set by a line, this implies $O(n^3 \log n)$ time to solve the connected two-center problem (by invoking the constrained connected two-center algorithm $O(n^2)$ times). However, by using the sweepline approach posed in [14] and the technique of randomized incremental construction of Voronoi diagram [10], step 2.1 can be completed in $O(\log n)$ time. Thus the total time complexity to solve the connected two-center problem is improved to $O(n^3)$.

Theorem 6 The connected two-center problem, for a set of n points in the plane, can be solved in $O(n^3)$ time.

5 Conclusions

We have shown the notion of the center-hull of a set of points in the plane and the relationship between the center-hull and the planar farthest-point Voronoi diagram. Because center-hulls have nice geometric properties, especially for the center problems, we believe that center-hulls will be very helpful in solving a large variety of geometric problems.

We also have presented an $O(n^3)$time algorithm for solving the connected two-center problem which is a variant of the two-center problem. Our algorithm could be improved if the number of linear partitions to be examined is reduced. Alternately, some unexploited properties of center-hulls will help us to design a better algorithm.

Like the two-center problem, the lower bound for the connected two-center problem is still open now. It is of interest to design approximation algorithms for solving connected k-center problems. Furthermore, of much great interest is the question of solving the connected two-center problem in three dimensions.

References

[1] P. Agarwal and M. Sharir, Planar geometric location problems, Algorithmica, 11(1994), pp.185–195.

[2] F. Chin and C.A. Wang, Optimal algorithms for intersection and minimum distance problems between planar polygons, IEEE Trans. Comput., vol. C-32, Dec. 1983, pp.1203–1207.

[3] Z. Drezner, Facility Location: A Survey of Applications and Methods, Springer Verlag, New York, 1995.

[4] Z. Drezner, The planer two-center and two-median problems, Transportation Science, 18(1984), pp.351–361.

[5] D.S. Dummit and R.M. Foote, Abstract Algebra, Prentice Hall, 1991, pp.538–549.

[6] H. Edelsbrunner, D.G. Kirkpatrick and R. Seidel, On the shape of a set of points in the plane, IEEE Trans. Inform. Theory, IT-29(1983), pp.551–559.

[7] H. Edelsbrunner, Algorithms in Combinatorial Geometry, EATCS Monographs on Theoretical Computer Science, V.10, Springer Verlag, New York, 1987.

[8] D. Eppstein, Dynamic three-dimensional linear programming, ORSA J. Computing, 4(1992), pp.360–368.

[9] D. Eppstein, Faster construction of planar two-centers, 8th ACM-SIAM Symp. Discrete Algorithms, 1997, pp.131–138.

[10] L.J. Guibas, D.E. Knuth and M. Sharir, Randomized Incremental Construction of Delaunay and Voronoi Diagrams, Algorithmica, 7(1992), pp.381-413.

[11] J. Hershberger, A faster algorithm for the two-center decision problem, Inform. Process. Lett., 47(1993), pp.23–29.

[12] J. Hershberger and S. Suri, Finding tailored partitions, J. Algorithms, 12(1991), pp.431–463.

[13] C.H. Huang, Some problems on radius-weighted model of packet radio network, Ph. D. Dissertation, Dept. of Comput. Sci., Tsing Hua Univ., Hsinchu, Taiwan, 1992.

[14] N.F. Huang, C.H. Huang and Y.L. Wang, A sweepline algorithm to solve the two-center problem, Inform. Process. Lett., 49(1994), pp.171-177.

[15] R.Z. Hwang, R.C.T. Lee and R.C. Chang, The slab dividing approach to solve the euclidean P-center problem, Algorithmica, 9(1993), pp.1-22.

[16] J. Jaromczyk and M. Kowaluk, An efficient algorithm for the Euclidean two-center problem, 10th ACM Symp. Computational Geometry, 1993, pp.303–311.

[17] M. Katz and M. Sharir, An expander-based approach to geometric optimization, 9th ACM Symp. Computational Geometry, 1993, pp.198–207.

[18] N. Megiddo, Linear-time algorithms for linear programming in R3 and related problems, SIAM J. Comput., 12(1983), pp.759–776.

[19] N. Megiddo and K. Supowit, On the complexity of some common geometric location problems, SIAM J. Comput., 13(1984), pp.1182–1196.

[20] F.P. Preparata and M.I. Shamos, Computational Geometry: An Introduction, Springer Verlag, New York, 1985.

[21] M.I. Shamos, Computational Geometry, Ph. D. Dissertation, Depr. of Comput. Sci., Yale Univ., New Haven, 1978.

[22] M. Sharir, A near-linear algorithm for the planar 2-center problem, 12th ACM Symp. Computational Geometry, 1996, pp.106-112.

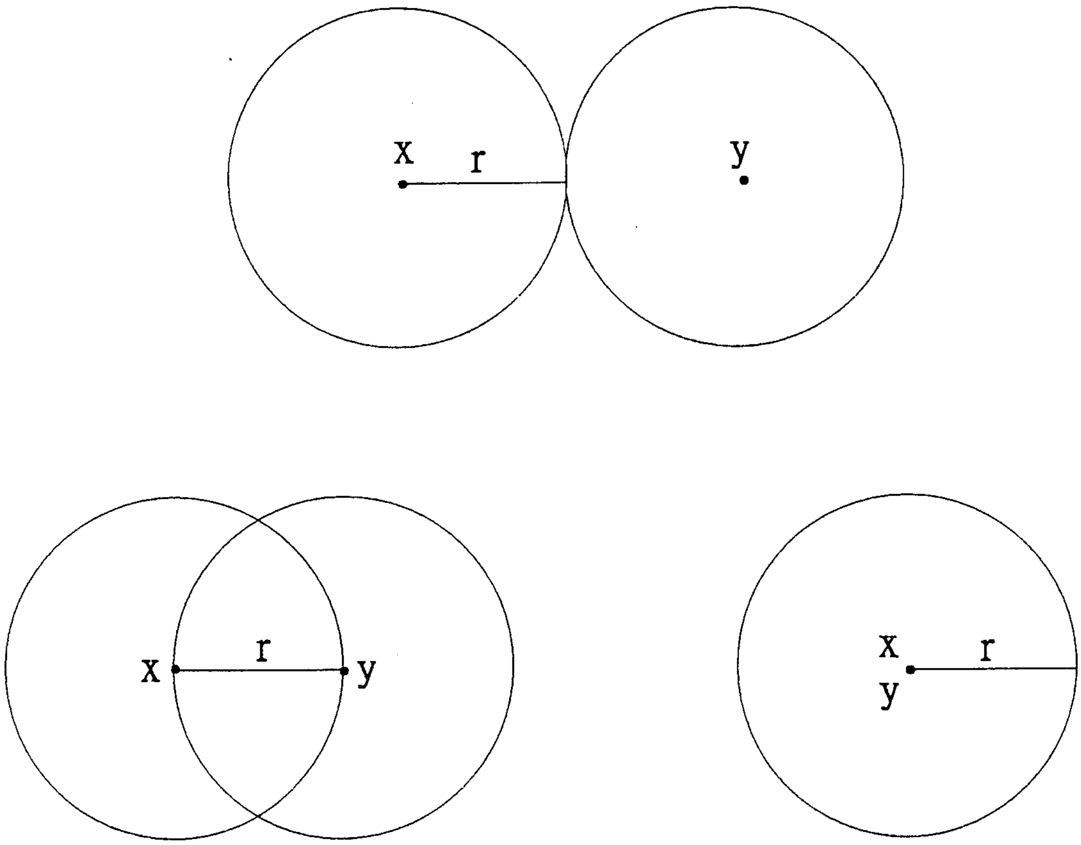

Figure 1: Three examples of α-C2C$(x, y, r)$with $\alpha = 0$, 1/2 and 1, respetcively.

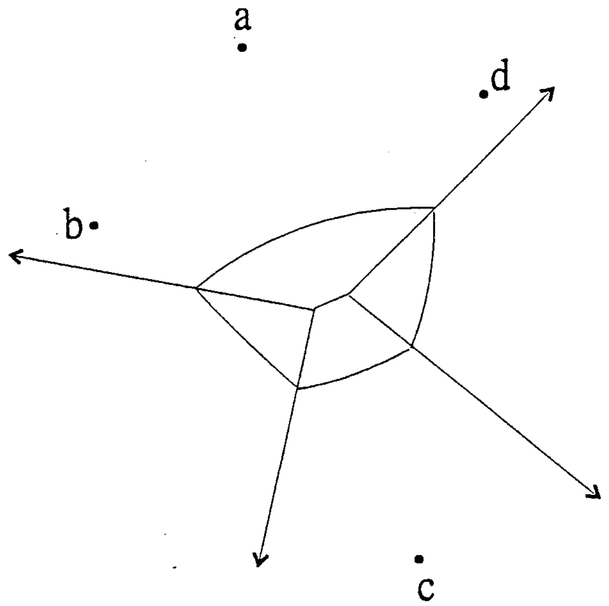

Figure 2: The farthest-point Voronoi diagram and the associated center-hull for the point set $N = \{a, b, c, d\}$.

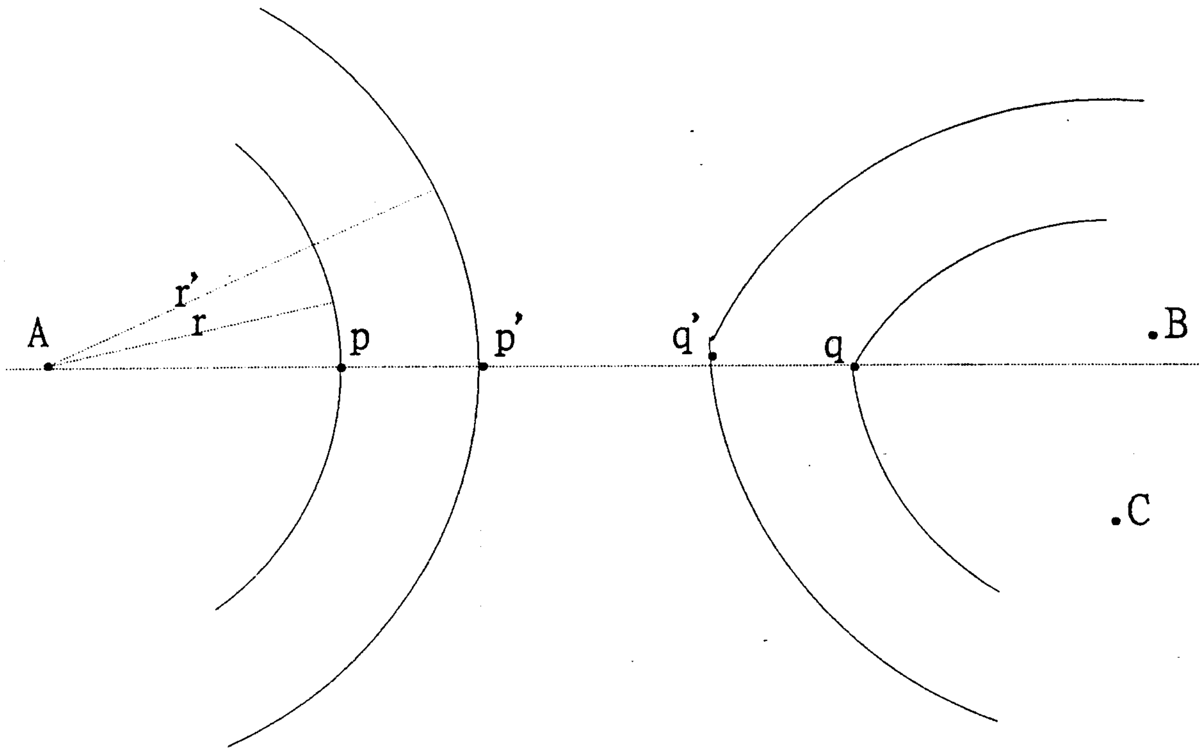

Figure 3:

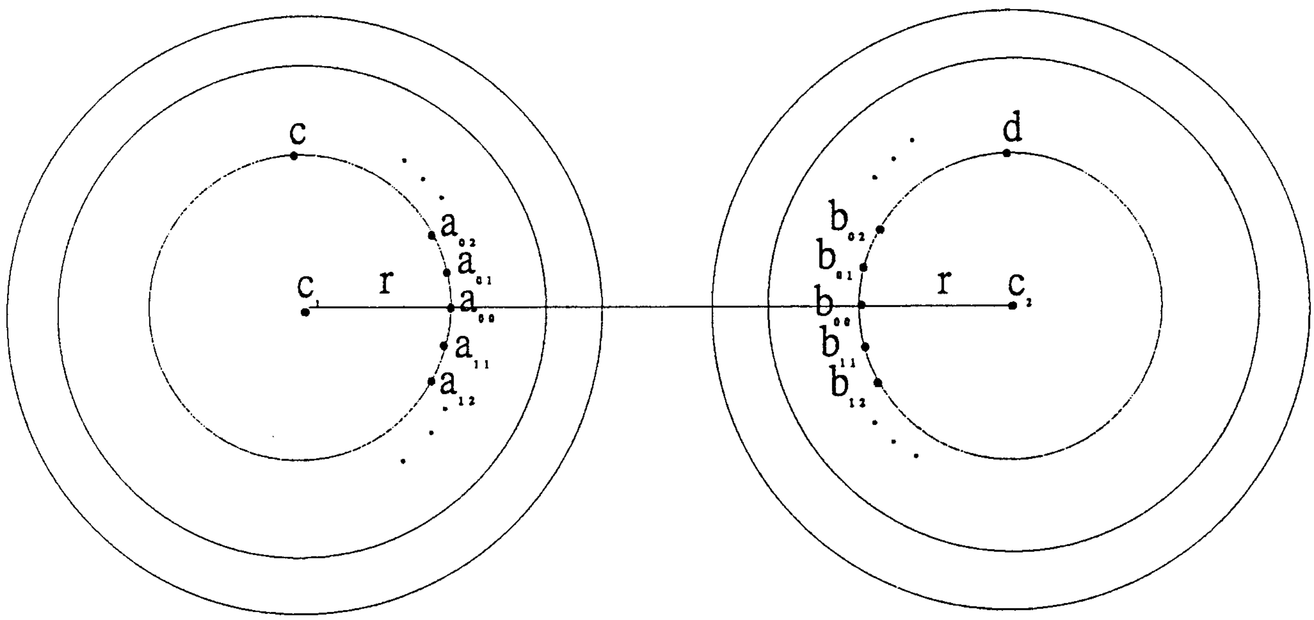

Figure 4: