FAULT-FREE RING EMBEDDING IN FAULTY WRAPPED BUTTERFLY GRAPHS *

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ABSTRACT

In this paper, we study cycle embedding in a faulty wrapped butterfly BF_n with at most two faults in vertices and/or edges. Let F be a subset of $V(BF_n) \cup E(BF_n)$ with $|F| \leq 2$. Let f_v denote $|F \cap V(BF_n)|$. In this paper, we prove that $BF_n - F$ contains a cycle of length $n \times 2^n - 2f_v$. Moreover, $BF_n - F$ contains a cycle of length $n \times 2^n - f_v$ if n is an odd integer. In other words, $BF_n - F$ contains a hamiltonian cycle if n is an odd integer.

1 INTRODUCTION

Performance of the distributed system is significantly determined by the choice of the network topology. The hypercube (binary n-cube) is one of the most popular interconnection networks. It has been used to design various commercial multiprocessor machines. One basic drawback with hypercubes is that the degree of nodes increases with the number of nodes. Hence it is not suitable to apply hypercubes to the area layout from the viewpoint of VLSI implementation. Among all networks of fixed degrees, wrapped butterfly network is one of the most promising networks due to its nice topological properties. On the other hand, cycle (ring) contains several attractive properties such as simplicity, extensibility, and feasible implementation. Hence embedding a cycle into wrapped butterfly network has received many researchers' efforts in recent years [1, 3, 5, 6, 8]. To embed a cycle into a faulty butterfly network, it is desirable to isolate those faulted components from the rest ones so that a maximal-length cycle can be still embedded.

Assume that $F \subset V(BF_n) \cup E(BF_n)$ be the fault set with $|F| \leq 2$. In [6], Vadapalli and Srimani verified that $BF_n - F$ contains a cycle of length $n \times 2^n - 2$ if F is a set with only one vertex and that $BF_n - F$ contains a cycle of length $n \times 2^n - 4$ if F is a set with two vertices. In [3], Hwang and Chen proved that there still exists a cycle of length $n \times 2^n$ in a $BF_n - F$ where F is a subset of $E(BF_n)$. In other words, $BF_n - F$ remains hamiltonian with at most two edges faults. In the previous study of cycle embedding into wrapped butterfly, faults are limited into either node faults or edge faults. However some faults on both nodes and edges may occur. Therefore we want to improve the results of [3, 6]. We use f_v to denote $|F \cap V(BF_n)|$. In this paper, we prove that $BF_n - F$ contains a cycle of length $n \times 2^n - 2f_v$. Moreover, $BF_n - F$ contains a cycle of length $n \times 2^n - f_v$ if n is an odd integer. In other words, $BF_n - F$ contains a hamiltonian cycle if n is an odd integer.

In the following section, we discuss some properties of the wrapped butterfly graphs. In section 3, we first present a short proof that $BF_n - F$ remains hamiltonian if F is a subset of $E(BF_n)$. Then we prove that $BF_n - F$ contains a cycle of length $n \times 2^n - 2$ if F is a set with one vertex and one edge. Finally, we prove that $BF_n - F$ contains a cycle of length $n \times 2^n - f_v$ if n is an odd integer.

2 WRAPPED BUTTERFLY AND ITS PROPERTIES

A *graph* $G = (V, E)$ consists of a finite set V and a subset E of $\{(u, v) \mid u \neq v,\ (u, v)$ is an unordered pair of elements of $V\}$. We call $V = V(G)$ the *vertex set* of G and $E = E(G)$ the *edge set* of G. Let $F = V_1 \bigcup E_1$ for $E_1 \subset E$ and $V_1 \subset V$. We use $G - F$ to denote the graph $G' = (V - V_1, (E - E_1) \bigcap ((V - V_1) \times (V - V_1)))$. The *wrapped butterfly* (*butterfly* for short) BF_n is a graph with $n \times 2^n$ vertices such that each vertex is labeled by $\langle a_0 a_1 \ldots a_{n-1}, i\rangle$ with $0 \leq i \leq n-1$ and $a_j \in \{0, 1\}$ for all $0 \leq j \leq n-1$. We say the vertex $\langle a_0 a_1 \ldots a_{n-1}, i\rangle$ is at *level* i. Edges of BF_n are described as follows. Node $\langle a_0 a_1 \ldots a_i \ldots a_{n-1}, i\rangle$ is adjacent to node $\langle a_0 a_1 \ldots a_i \ldots a_{n-1}, (i+1) \textbf{ mod } n\rangle$ by a *straight edge* and adjacent to node $\langle a_0 a_1 \ldots \bar{a}_i \ldots a_{n-1}, (i+1) \textbf{ mod } n\rangle$ by a *cross edge*.

Lemma 1 *[4]*
For any integer k with $0 \leq k < n$, the mapping σ_k from $V(BF_n)$ into $V(BF_n)$ defined by $\sigma_k(\langle a_0 a_1 \ldots a_{n-1}, l\rangle) = \langle a_k a_{k+1} \ldots a_{n-1} a_0 a_1 \ldots a_{k-1}, (l-k) \textbf{ mod } n\rangle$ is an automorphism of BF_n.

Similarly, we can easily obtain the following lemma.

Lemma 2 *For any integer i with $0 \leq i < n$, the mapping φ_i from $V(BF_n)$ into $V(BF_n)$ defined by*

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$\varphi_i(\langle a_0a_1\ldots a_{n-1},l\rangle)=\langle a_0a_1\ldots\bar{a}_ia_{i+1}\ldots a_{n-1},l\rangle$ *is an automorphism of* BF_n.

Thus, we have the following corollary.

Corollary 1 *BF_n is vertex transitive.*

In [5], Vadapalli et al. proposed a family of degree four Cayley graphs, G_n. Later, Chen and Lau [2] point out that G_n is isomorphic to BF_n. Thus, we can combine all the results of G_n and BF_n.

Each vertex of G_n is represented by a circular permutation of n different symbols in lexicographic order, where the n symbols are presented in either uncomplemented or complemented form. Let d_k, $0\le k\le n-1$, denote the kth symbol in the set of n symbols. We use the English alphabets: thus for $n=3$, $d_0=a$, $d_1=b$, and $d_2=c$. We use t_k to denote either d_k or $\bar{d}_k$. Therefore, for n distinct symbols, there are exactly n different cyclic permutations of the symbols in lexicographic order. Moreover, each symbol can be presented in either uncomplemented or complemented form. So the vertex set of G_n has a cardinality of $n\times 2^n$. If $a_0a_1\ldots a_{n-1}$ denotes the label of an arbitrary vertex and $a_0=t_k$ for some integer k, then for all i and $0\le i\le n-1$, we have $a_i=t_l$ where $l=k+i$ (**mod** n). The edges of G_n are defined by the following four generators in the graph:

$$\begin{aligned}
&g(t_kt_{k+1}\ldots t_{n-1}t_0\ldots t_{k-2}t_{k-1})\\
&=t_{k+1}\ldots t_{n-1}t_0\ldots t_{k-1}t_k,\\
&f(t_kt_{k+1}\ldots t_{n-1}t_0\ldots t_{k-2}t_{k-1})\\
&=t_{k+1}\ldots t_{n-1}t_0\ldots t_{k-1}\bar{t}_k,\\
&g^{-1}(t_kt_{k+1}\ldots t_{n-1}t_0\ldots t_{k-2}t_{k-1})\\
&=t_{k-1}t_k\ldots t_{n-1}t_0\ldots t_{k-2},\ \text{and}\\
&f^{-1}(t_kt_{k+1}\ldots t_{n-1}t_0\ldots t_{k-2}t_{k-1})\\
&=\bar{t}_{k-1}t_k\ldots t_{n-1}t_0\ldots t_{k-2}.
\end{aligned}$$

In [7], Wei et al. point out the isomorphism maps the vertex $\langle a_0a_1\ldots a_{n-1},k\rangle$ of BF_n into the vertex $t_k\ldots t_{n-1}t_0\ldots t_{k-1}$ of G_n, where $t_i=d_i$ if and only if $a_i=0$, or $t_i=\bar{d}_i$ if and only if $a_i=1$. Therefore, throughout this paper, the nodes of the butterfly graph will be labeled in the form of $\langle a_0a_1\ldots a_{n-1},k\rangle$ rather than $t_k\ldots t_{n-1}t_0\ldots t_{k-1}$. Therefore, the four generators g, g^{-1}, f and f^{-1} can be rewritten as follows:

$$\begin{aligned}
&g(\langle a_0a_1\ldots a_{n-1},k\rangle)\\
&=\langle a_0a_1\ldots a_{n-1},k+1\rangle,\\
&f(\langle a_0a_1\ldots a_{n-1},k\rangle)\\
&=\langle a_0a_1\ldots a_{k-1}\bar{a}_ka_{k+1}\ldots a_{n-1},k+1\rangle,\\
&g^{-1}(\langle a_0a_1\ldots a_{n-1},k\rangle)\\
&=\langle a_0a_1\ldots a_{n-1},k-1\rangle,\ \text{and}\\
&f^{-1}(\langle a_0a_1\ldots a_{n-1},k\rangle)\\
&=\langle a_0a_1\ldots a_{k-2}\bar{a}_{k-1}a_k\ldots a_{n-1},k-1\rangle.
\end{aligned}$$

Hence the *g-edges*, $(u,g(u))$ or $\big(u,g^{-1}(u)\big)$ for some $u\in V(BF_n)$, correspond to the straight edges and the *f-edges*, $(u,f(u))$ or $\big(u,f^{-1}(u)\big)$ for some $u\in V(BF_n)$, correspond to the cross edges of BF_n.

Lemma 3 *$f^{-1}(g(u))=g^{-1}(f(u))$ for any node u in BF_n.*

Let u be any vertex of BF_n. Obviously, $g^n(u)=u$. Moreover, $\langle u,g(u),g^2(u),\ldots,g^n(u)=u\rangle$ forms a simple cycle of length n, denoted by C_g^u. We call such cycle of BF_n a *g-cycle* at u. It is easy to see that $C_g^v=C_g^u$ if and only if $v\in C_g^u$. Thus all g-cycles form a partition of the straight edges of BF_n. There is no g-edge joining vertices of two different g-cycles. Any f-edge joins vertices of two different g-cycles. Obviously, $(u,f(u))$ joins vertices of C_g^u and $C_g^{f(u)}$. The following lemma can be proved easily.

Lemma 4 *$\big(g(u),g^{-1}(f(u))\big)$ is an f-edge joining vertices of C_g^u and $C_g^{f(u)}$. Moreover, the path $\langle u,f(u),g^{-1}(f(u)),g(u),u\rangle$ forms a cycle of length 4.*

Any C_g^u contains exactly one vertex at each level. In particular, C_g^u contains exactly one vertex at level 0, say $\langle a_0a_1\ldots a_{n-1},0\rangle$. We use $C_g^{(a_0a_1\ldots a_{n-1})}$ as the name for C_g^u. Now, we form a new graph BF_n^G with all the g-cycles of BF_n as vertices, two different g-cycles are joined with an edge if and only if there exists an f-edge joining them. The vertex of BF_n^G corresponding to C_g^u is denoted by $\bar{C}_g^u$. The following theorem is proved in [5] [6].

Lemma 5 *BF_n^G is isomorphic to the n-dimensional hypercube. Moreover, the set of vertices adjacent to the vertex corresponding to $C_g^{(a_0a_1\ldots a_{n-1})}$ is the set of vertices corresponding to the g-cycles in $\{C_g^{(\bar{a}_0a_1\ldots a_{n-1})},C_g^{(a_0\bar{a}_1\ldots a_{n-1})},\ldots C_g^{(a_0a_1\ldots\bar{a}_{n-1})}\}$.*

Let $h=(\bar{C}_g^u,\bar{C}_g^v)$ be any edge of BF_n^G. We use $X(h)$ to denote the set of edges in BF_n joining vertices of C_g^u and C_g^v. Using standard counting technique, we have the following two corollaries.

Corollary 2 *Let $h=(\bar{C}_g^u,\bar{C}_g^v)$ be any edge of BF_n^G. Then $|X(h)|=2$. Moreover, the vertices of edges in $X(h)$ induces a 4-cycle in BF_n.*

Corollary 3 *There is a unique g-cycle, namely $C_g^{f(u)}$, such that edges of BF_n joining vertices between C_g^u and $C_g^{f(u)}$ are exactly $(u,f(u))$ and $\big(g(u),f^{-1}(g(u))\big)$.*

According to Corollaries 2 and 3, any edge $h=(\bar{C}_g^u,\bar{C}_g^v)$ in BF_n^G induces a unique 4-cycle in BF_n, with two f-edges and two g-edges. We use $X_f(C_g^u,C_g^v)$ to denote the set of f-edges in this 4-cycle, and $X_g(\bar{C}_g^u,\bar{C}_g^v)$ to denote the set of g-edges in this cycle.

Lemma 6 *Assume that T be any subtree of BF_n^G. Let C_g^T denote the graph generated by the edge set*

$$\left(\bigcup_{\bar{C}_g^u \in V(T)} E(C_g^u) \cup \bigcup_{(\bar{C}_g^u, \bar{C}_g^v) \in E(T)} X_f(C_g^u, C_g^v)\right) - \bigcup_{(\bar{C}_g^u, \bar{C}_g^v) \in E(T)} X_g(C_g^u, C_g^v).$$

Then C_g^T is a cycle of BF_n of length $n \times |V(T)|$.

Let $u = \langle a_0 a_1 \ldots a_{n-1}, k\rangle$ be any vertex of BF_n. We use $\tilde{u}$ to denote the node $\langle \bar{a}_0 \bar{a}_1 \ldots \bar{a}_{n-1}, k\rangle$. Obviously, $f^n(u) = \tilde{u}$ and $f^{2n}(u) = u$. Moreover, $\langle u, f(u), f^2(u), \ldots, f^{2n}(u) = u\rangle$ forms a simple cycle of length $2n$, denoted by C_f^u. It is easy to see that all f-cycles form a partition of the cross edges of BF_n. There is no f-edge joining vertices of two different f-cycles. Any g-edge joins vertices of two different f-cycles. The g-edge $(u, g(u))$ joins vertices of C_f^u and $C_f^{g(u)}$. The following lemma can be proved easily.

Lemma 7 $(f(u), g^{-1}(f(u)))$, $(\tilde{u}, g(\tilde{u}))$, $(f(\tilde{u}), g^{-1}(f(\tilde{u})))$ *are also g-edges joining vertices of C_f^u and $C_f^{g(u)}$. Moreover, the paths $\langle u, f(u), g^{-1}(f(u)), g(u), u\rangle$, and $\langle \tilde{u}, f(\tilde{u}), g^{-1}(f(\tilde{u})), g(\tilde{u}), \tilde{u}\rangle$, form two 4-cycles in BF_n.*

Any C_f^u contains exactly two vertex at each level. Suppose that u is one of the vertex in C_f^u at level i. Obviously, the other vertex in C_f^u at level i is $\tilde{u}$. Thus, C_f^u contains exactly one vertex at level 0, say $\langle a_0 a_1 \ldots a_{n-1}, 0\rangle$ with $a_{n-1} = 0$. We use $C_f^{(a_0 a_1 \ldots a_{n-2})}$ as the name for C_g^u. Now, we form a new graph BF_n^F with all the f-cycles of BF_n as vertices, two different f-cycles are joined with an edge if and only if there exists a g-edge joining them. The vertex of BF_n^F corresponding to C_f^u is denoted by $\bar{C}_f^u$. The following theorem is proved in [5] [6].

Lemma 8 *BF_n^F is isomorphic to the $(n-1)$-dimensional folded hypercube. Moreover, the set of vertices adjacent to the vertex corresponding to $C_f^{(a_0 a_1 \ldots a_{n-2})}$ is the set of vertices corresponding to the f-cycles in* $\{C_f^{(\bar{a}_0 a_1 \ldots a_{n-2})}, C_f^{(a_0 \bar{a}_1 \ldots a_{n-2})}, \ldots C_f^{(a_0 a_1 \ldots \bar{a}_{n-2})}\} \cup \{C_f^{(\bar{a}_0 \bar{a}_1 \ldots \bar{a}_{n-2})}\}$.

Let $h = (\bar{C}_f^u, \bar{C}_f^v)$ be any edge of BF_n^F. We use $Y(h)$ to denote the set of edges of BF_n joining vertices of C_f^u and C_f^v. Using standard counting technique, we have the following two corollaries.

Corollary 4 *Let $h = (\bar{C}_f^u, \bar{C}_f^v)$ be any edge of BF_n^F. Then $|Y(h)| = 4$. Moreover, the vertices of edges in $Y(h)$ induce two 4-cycles in BF_n.*

Corollary 5 *There is a unique f-cycle, namely $C_f^{g(u)}$, such that edges of BF_n joining vertices between C_f^u and $C_f^{g(u)}$ are exactly $(u, g(u))$, $\left(f(u), g^{-1}(f(u))\right)$, $(\tilde{u}, g(\tilde{u}))$, and $\left(f(\tilde{u}), g^{-1}(f(\tilde{u}))\right)$.*

According to Corollaries 4 and 5, any edge $h = (\bar{C}_f^u, \bar{C}_f^v)$ induces two 4-cycles in BF_n. Let α be an assignment of $(\bar{C}_f^u, \bar{C}_f^v) \in E(BF_n^F)$ with one of the 4-cycles it induced. We use $Y_f^\alpha(C_f^u, C_f^v)$ to denote the set of f-edges induced by $\alpha(h)$ and $Y_g^\alpha(C_f^u, C_f^v)$ to denote the set of g-edges induced by $\alpha(h)$. Hence $|Y_f^\alpha(C_f^u, C_f^v)| = |Y_g^\alpha(C_f^u, C_f^v)| = 2$.

Lemma 9 *Assume that T is any subset of BF_n^F. Let $C_f^{T,\alpha}$ denote the graph generated by the edge set*

$$\left(\bigcup_{\bar{C}_f^u \in V(T)} E(C_f^u) \cup \bigcup_{(\bar{C}_f^u, \bar{C}_f^v) \in E(T)} Y_g^\alpha(C_f^u, C_f^v)\right) - \bigcup_{(\bar{C}_f^u, \bar{C}_f^v) \in E(T)} Y_f^\alpha(C_f^u, C_f^v).$$

Then $C_f^{T,\alpha}$ is a cycle of BF_n of length $2n \times |V(T)|$.

In the following, we introduce three basic cycles $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$.

The cycle $\mathcal{B}_1$ is constructed as follows: Let $a_1 = \langle \underbrace{00\ldots0}_{n}, 1\rangle$. Let P_1 be the path a_1, $g(a_1)$, …, $g^{n-2}(a_1) = a_2$. Obviously, $a_2 = \langle \underbrace{00\ldots0}_{n}, n-1\rangle$, $f(a_2) = \langle \underbrace{00\ldots0}_{n-1} 1, 0\rangle = a_3$, and $f(a_3) = \langle 1 \underbrace{00\ldots0}_{n-2} 1, 1\rangle = a_4$. Let P_2 be the path a_4, $g(a_4)$, …, $g^{n-1}(a_4) = a_5$. Obviously, $a_5 = \langle 1 \underbrace{00\ldots0}_{n-2} 1, 0\rangle$ and $f(a_5) = \langle 1 \underbrace{00\ldots0}_{n-1}, n-1\rangle = a_6$. Let P_3 be the path a_6, $g^{-1}(a_6)$, …, $g^{-(n-1)}(a_6) = a_7$. Obviously, $a_7 = \langle 1 \underbrace{00\ldots0}_{n-1}, 0\rangle$ and $f(a_7) = a_1$. Then $\mathcal{B}_1$ is $\langle a_1 \to P_1 \to a_2, a_3, a_4 \to P_2 \to a_5, a_6 \to P_3 \to a_7, a_1\rangle$. Let $W_1 = V(C_g^{a_1}) \cup V(C_g^{a_3}) \cup V(C_g^{a_5}) \cup V(C_g^{a_7})$ and $\bar{W}_1 = \{\bar{C}_g^{a_1}, \bar{C}_g^{a_3}, \bar{C}_g^{a_5}, \bar{C}_g^{a_7}\}$.

The cycle $\mathcal{B}_2$ is constructed as follows: Let $b_1 = \langle \underbrace{00\ldots0}_{n}, 1\rangle$. Let Q_1 be the path b_1, $g(b_1)$, …, $g^{n-2}(b_1) = b_2$. Obviously, $b_2 = \langle \underbrace{00\ldots0}_{n}, n-1\rangle$ and $f^{-1}(b_2) = \langle \underbrace{00\ldots0}_{n-2} 10, n-2\rangle = b_3$. Let Q_2 be the path b_3, $g^{-1}(b_3)$, …, $g^{-(n-3)}(b_3) = b_4$. Obviously, $b_4 = \langle \underbrace{00\ldots0}_{n-2} 10, 1\rangle$ and $f^{-1}(b_4) = \langle 1 \underbrace{00\ldots0}_{n-3} 10, 0\rangle = b_5$. Let Q_3 be the path b_5, $g(b_5)$, …, $g^{n-1}(b_5) = b_6$. Obviously, $b_6 = \langle 1 \underbrace{00\ldots0}_{n-3} 10, n-1\rangle$ and $f^{-1}(b_6) = \langle 1 \underbrace{00\ldots0}_{n-1}, n-2\rangle = b_7$.

Let Q_4 be the path $b_7, g(b_7), g^2(b_7) = b_8$. Then $b_8 = \langle 1\underbrace{00\ldots0}_{n-1}, 0\rangle$ and $f(b_8) = \langle\underbrace{00\ldots0}_{n}, 1\rangle = b_1$. Then $\mathcal{B}_2$ is $\langle b_1 \to Q_1 \to b_2, b_3 \to Q_2 \to b_4, b_5 \to Q_3 \to b_6, b_7 \to Q_4 \to b_8, b_1\rangle$. Let $W_2 = V(C_g^{b_1}) \cup V(C_g^{b_3}) \cup V(C_g^{b_5}) \cup V(C_g^{b_7})$ and $\bar{W}_2 = \{\bar{C}_g^{b_1}, \bar{C}_g^{b_3}, \bar{C}_g^{b_5}, \bar{C}_g^{b_7}\}$.

Let $2 \leq j \leq d-1$. The cycle $\mathcal{B}_3$ is constructed as follows: Let $c_1 = \langle\underbrace{00\ldots0}_{n}, 1\rangle$. Let R_1 be the path c_1, $g(c_1)$, …, $g^{j-2}(c_1) = c_2$. Obviously, $c_2 = \langle\underbrace{00\ldots0}_{n}, j-1\rangle$ and $f(c_2) = \langle\underbrace{00\ldots0}_{j-1}1\underbrace{00\ldots0}_{n-j}, j\rangle = c_3$. Let R_2 be the path c_3, $g^{-1}(c_3)$, …, $g^{-j}(c_3) = c_4$. Obviously, $c_4 = \langle\underbrace{00\ldots0}_{j-1}1\underbrace{00\ldots0}_{n-j}, 0\rangle$ and $f(c_4) = \langle 1\underbrace{00\ldots0}_{j-2}1\underbrace{00\ldots0}_{n-j}, 1\rangle = c_5$. Let R_3 be the path c_5, $g^{-1}(c_5)$, …, $g^{-(n-j+1)}(c_5) = c_6$. Obviously, $c_6 = \langle 1\underbrace{00\ldots0}_{j-2}1\underbrace{00\ldots0}_{n-j}, j\rangle$ and $f^{-1}(c_6) = \langle 1\underbrace{00\ldots0}_{n-1}, j-1\rangle = c_7$. Let R_4 be the path $c_7, g^{-1}(c_7), \ldots, g^{-(n-j+1)}(c_7) = c_8$. Then $c_8 = \langle 1\underbrace{00\ldots0}_{n-1}, 0\rangle$ and $f(c_8) = \langle\underbrace{00\ldots0}_{n}, 1\rangle = c_1$. Then $\mathcal{B}_3$ is $\langle c_1 \to R_1 \to c_2, c_3 \to R_2 \to c_4, c_5 \to R_3 \to c_6, c_7 \to R_4 \to c_8, c_1\rangle$. Then, the length of $\mathcal{B}_3$ is $2n+4$. Let $W_3 = V(C_g^{c_1}) \cup V(C_g^{c_3}) \cup V(C_g^{c_5}) \cup V(C_g^{c_7})$ and $\bar{W}_3 = \{\bar{C}_g^{c_1}, \bar{C}_g^{c_3}, \bar{C}_g^{c_5}, \bar{C}_g^{c_7}\}$.

When $n = 3$, it is observed that $b_3 = b_4$ and $c_1 = c_2$. All the vertices of $\mathcal{B}_i$ is a proper subset of W_i for every $1 \leq i \leq 3$. Moreover, the length of $\mathcal{B}_i$ is $3n$ for $i = 1, 2$.

3 CYCLE EMBEDDING IN A FAULTY WRAPPED BUTTERFLY

In this section, we assume that $F \subset V(BF_n) \bigcup E(BF_n)$ with $|F| \leq 2$. In the following lemmas, we just state the results and omit the proofs.

Lemma 10 *For any integer n with $n \geq 3$, $BF_n - F$ is hamiltonian if $F \subset E(BF_n)$ and $|F| = 2$.*

Lemma 11 *Assume that $n \geq 3$. Then $BF_n - F$ contains a cycle of length $n \times 2^n - 2$ where F consists of a vertex and an edge in BF_n.*

Lemma 12 *For any odd integer n with $n \geq 3$, $BF_n - F$ is hamiltonian where F consists of a vertex and an edge in BF_n.*

Lemma 13 *For any odd integer n with $n \geq 3$, $BF_n - F$ is hamiltonian where $F \subset V(BF_n)$ and $|F| = 2$.*

Since BF_n is hamiltonian for all $n \geq 3$, by lemmas 10, 11, 12, 13, and Vadapalli et. al. [6], we have the following theorem.

Theorem 1 *Assume that $n \geq 3$, $F \subset V(BF_n) \bigcup E(BF_n)$, and $|F| \leq 2$. Then $BF_n - F$ contains a cycle of length $n \times 2^n - 2|F \cap V(BF_n)|$. Moreover, $BF_n - F$ contains a hamiltonian cycle if n is an odd integer.*

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