AN EFFICIENT ALGORITHM FOR TRANSVERSAL OF DISJOINT CONVEX POLYGONS[Ψ]

Francis Y.L. Chin[1], *Hong Shen*[2], *Fu Lee Wang*[3]

1. Department of Computer Science and Information Systems
The University of Hong Kong, Pokfulam Road, Hong Kong
Email: chin@csis.hku.hk

2. School of Computing and Information Technology
Griffith University, Queensland, Australia
Email: hong@cit.gu.edu.au

3. Department of Systems Engineering and Engineering Management
The Chinese University of Hong Kong, Shatin, Hong Kong
Email: flwang@se.cuhk.edu.hk

ABSTRACT

Given a set $\boldsymbol{S}$ of n disjoint convex polygons $\{P_i \mid 1 \leq i \leq n\}$ in a plane, each with k_i vertices, the *transversal problem* is to determine whether there exists a straight line that goes through every polygon in $\boldsymbol{S}$. We show that the transversal problem can be solved in $O(N + n \log n)$ time, where $N=\sum_{i=1}^{n} k_i$ is the total number of vertices of the polygons.

1. INTRODUCTION

Given a set $\boldsymbol{S}$ of n objects, a *transversal* or *stabber* is a straight line which intersects every member of $\boldsymbol{S}$. Edelsbrunner et al [7] gave an $O(n \log n)$ algorithm to construct a transversal of n line segments. If $\boldsymbol{S}$ consists of n parallel line segments, then a transversal can be found in $O(n)$ time by reducing the problem to a 2-variable linear programming problem with $2n$ constraints [5, 8, 9]. In [6], Edelsbrunner extended the linear programming method to show that a transversal of n rectangles can also be found in $O(n)$ time, and a transversal of n homothets[ξ] in $O(n \log n)$ time. In fact, it has been shown [3] that there exists an $\Omega(n \log n)$ lower bound for finding a transversal of n homothets (even for translates) of a circle.

Let $S = \{P_1, \ldots, P_n\}$, P_i be a convex polygon of k_i vertices, and $N = \sum_{i=1}^{n} k_i$ be the total number of vertices of these n convex polygons. The transversal of n polygons was studied in [2]. Two vertices p_1 and p_2 of a polygon P are an *antipodal pair* if there exist two parallel lines supporting P at p_1 and p_2. By finding all the antipodal pairs in n polygons, arranging them according their orientations and solving N transversal problems of n line segments, the transversal of n polygons can be found in $O(nN + N \log N)$ time. In [1], the transversal problem of n convex k-gons was studied and solved in $O(N \log N\, \alpha(n))$ time where $N = kn$ and $\alpha(n)$ is the inverse of the Ackermann function.

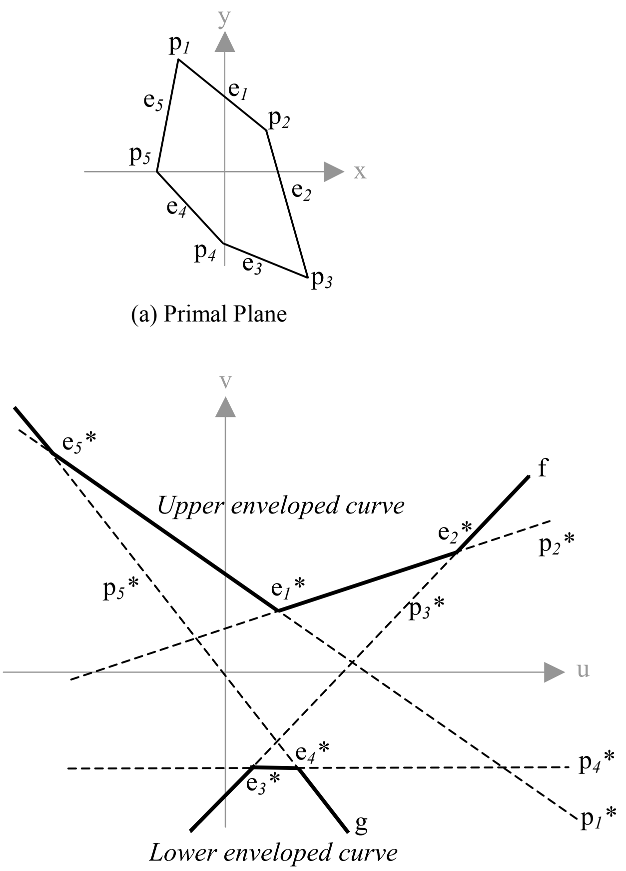
Ψ This research is supported by RGC Research Grant HKU 7024/98E.

ξ An object h is called a *homothet* of another object g if there exists a vector v and a positive real number m such that $h = mg + v$. In other words, object h is a translate of another object with the same shape as object g.

In this paper, we shall use an approach similar to the one given in [1] to solve the transversal problem of n disjoint polygons with a total of N vertices. With a new trick and careful analysis of the algorithm, we show that this transversal problem can be solved in $O(N + n \log n)$ time.

2. UPPER AND LOWER ENVELOPED CURVES

The *primal-dual* transformation [4] is applied to solve this transversal problem. A point $p = (a, b)$ in the *primal plane* will be transformed to a dual line $p^* = \{(u,v) \mid v = au + b\}$ in the *dual plane*, similarly a line $l = \{(x, y) \mid y = mx + c\}$ in the primal plane to a dual point $l^* = (-m, c)$ in the dual plane. A convex polygon is transformed to two polygonal curves, an upper enveloped curve f and a lower enveloped curve g in the dual plane as given in Figure 1. It is easy to show that a convex polygon P with k vertices has two polygonal curves in the dual plane with a total of $k+2$ edges and k vertices.



(a) Primal Plane

(b) Dual Plane

Figure 1. *A Polygon in Primal Plane and its enveloped curves in Dual Plane*

A stabbing line of the polygon in the primal plane corresponds to a dual point that lies within the strip bounded by the two enveloped polygonal curves in the dual plane. Thus the stabbing line to a set of n polygons can be found by finding the intersection of n regions bounded by their pairs of enveloped curves, in other words, the regions bounded by the minima curve $U_n(u)$ of the upper enveloped curves $\{f_i(u) \mid 1 \le i \le n\}$ and the maxima curve $V_n(u)$ of the lower enveloped curves $\{_i(u) \mid 1 \le i \le n\}$, i.e., the region between $U_n(u) = \min_{1 \le i \le n}\{f_i(u)\}$ and $V_n(u) = \max_{1 \le i \le n}\{g_i(u)\}$. Thus, for any u, if the minima curve is vertically above the maxima curve, i.e. $U_n(u) \ge V_n(u)$, then there exist stabbing lines for this set of polygons. From the definitions of $U_n(u)$ and $V_n(n)$, $U_n(u)$ and $V_n(u)$ are formed by edges of the upper and lower enveloped curves respectively.

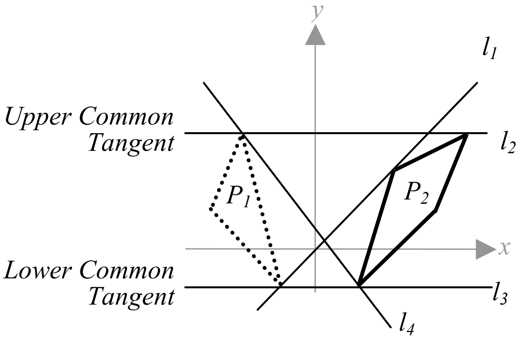
Let u^+ denote the half-plane of $u \ge 0$ and u^- the half-plane of $u < 0$ in the dual plane. The following lemmas prove the properties of $U_n(u)$ and $V_n(u)$.

Lemma 1: *Given two disjoint convex polygons, P_1 and P_2. Let f_i and g_i be the upper and lower enveloped curves of P_i where i = 1 and 2. Then f_1 and f_2, g_1 and g_2 will intersect at most one point in u^+ and at most one point in u^-.*

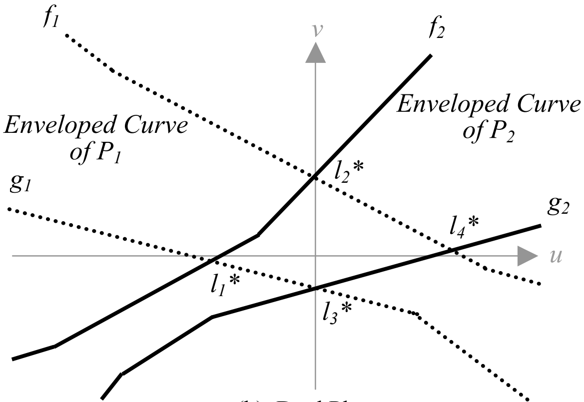
Proof: Between any two disjoint convex polygons, there are exactly four common tangents of these two polygons (Figure 2), which are represented by the four intersection points of the enveloped curves.

Let us consider the outer common tangents only, i.e., l_2 and l_3. A common tangent is called the *upper (lower) common tangent* if there exist no parts of the polygons lying above (below) the tangent. As given in Figure 2, there are normally one upper common tangent and one lower common tangent represented by the intersections of f_1 with f_2 and g_1 with g_2 in the dual plane, respectively. Thus, f_1 intersects f_2 and g_1 intersects g_2 exactly at one point either in u^+ or u^-. The claim of the lemma holds trivially.

However, in some situations, e.g., when two polygons are placed one on top of the other, the two outer common tangents can be both upper or both lower (Figures 3 and 4).

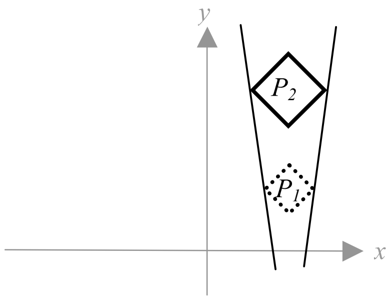


(a) Primal Plane

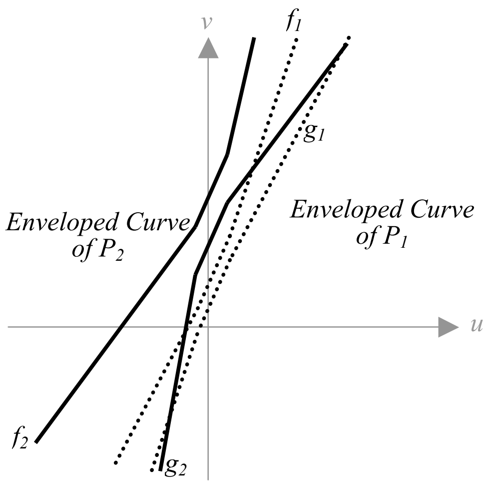


(b) Dual Plane

Figure 2. *Two Polygons and Their Enveloped Curves in the Dual Plane*

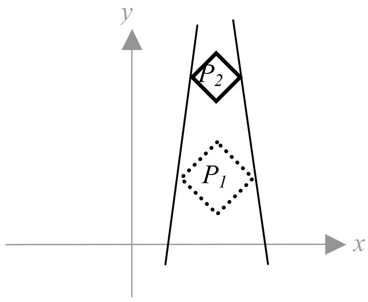


(a) Primal Plane

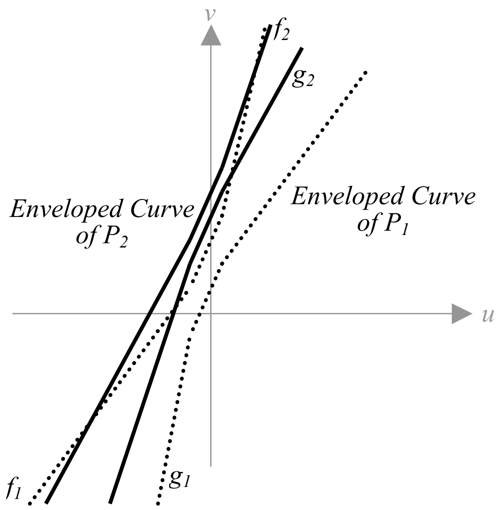


(b) Dual Plane

Figure 3. *Two Vertically-placed Polygons with Two Lower Common Outer Tangents and Their Enveloped Curves in Dual Plane*



(a) Primal Plane



(b) Dual Plane

Figure 4. *Two Vertically-placed Polygons with Two Upper Common Outer Tangents and Their Enveloped Curves in Dual Plane*

In either case, one of the outer common tangents is with a positive slope (i.e., $u \geq 0$) and the other with a negative slope (i.e., $u < 0$). Thus, one of the intersection points must lie in u^+ and the other in u^-. Thus, the lemma is also proved.

Although every two upper (lower) enveloped curves must intersect each other at most one point in u^+ and at most one in u^- (Lemma 1), the following lemma (Lemma 2) shows that the minima (maxima) curve of n upper (lower) enveloped curves does not necessarily have $O(n^2)$ vertices/edges. As the algorithms of the minima and maxima curves are similar, only the algorithm for the minima curve $U_n(u)$ will be discussed in this section.

Lemma 2: *Given a set of n disjoint convex polygons with a total of N vertices, the minima(maxima) curve of their upper(lower) enveloped curves in the dual plane has at most O(N) edges and vertices.*

Proof: Given a set of n disjoint convex polygons, the vertices in the minima curve in the dual plane must come from either the vertices of the upper enveloped curves or the intersection points of the upper enveloped curves. Consider an intersection point in the minima curve, it represents an upper common outer tangent of two polygons in the primal plane (Figure 5). Let us define such tangent as an *upper limiting line*. As there are normally at most two upper limiting lines (one on the left and one on the right) associated with each polygon, there exist at most $2n$ intersection points of enveloped curves in the minima curve. If the polygons are stacked up one above the other, each polygon might associate with four upper limiting lines, two having positive slope and the other two having negative slopes (Lemma 1). Since there exist a total of O(N) vertices in the upper enveloped curves, and $N > n$, there are at most O(N) vertices in the minima curve.

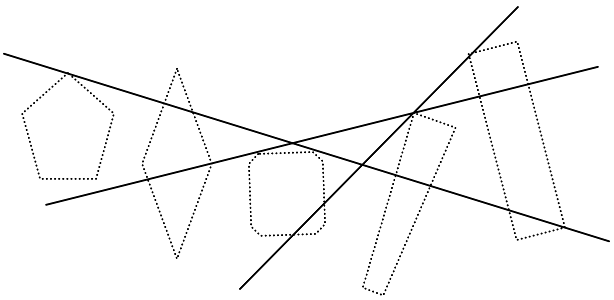


Figure 5. *Upper Limiting Lines of a Set of Polygons*

3. THE ALGORITHM

The minima curve of the upper enveloped curves can be constructed incrementally. The enveloped curves in u^- are added according to the decreasing order of their values at negative infinity (or their slopes at negative infinity), i.e., $f_1(u) > f_2(u) > \ldots > f_n(u)$ when $u \to -\infty$. Assume U_m is the minima curve formed by the first m upper enveloped curves, i.e., $U_m(u) = \min_{1 \le i \le n}\{f_i(u)\}$. U_{m+1} in u^- is formed by sweeping U_m and f_{m+1} from left to right until they intersect or $u = 0$. If U_m and f_{m+1} intersect in u^-, then U_{m+1} is formed by replacing the left part of U_m by the left part of f_{m+1} (Figure 6), otherwise, replacing the whole U_m by f_{m+1}. U_{m+1} in u^+ is formed similarly by inserting the enveloped curves one by one according to the descending order of their values at positive infinity (or their slopes at positive infinity), and the sweeping is performed from right to left until $u = 0$.

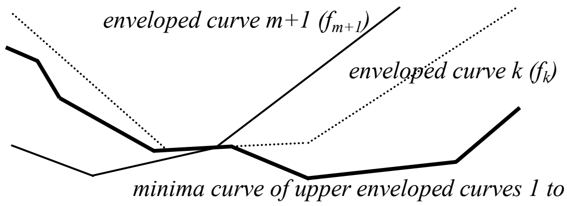


Figure 6. *Construction of U_{m+1} in u^- from f_{m+1} and U_m*

Lemma 3: *The upper(lower) enveloped curve $f_{m+1}(g_{m+1})$ will intersect the minima(maxima) curve $U_m(V_m)$ at most one point in u^+ and at most one point in u^-.*

Proof: When sweeping the functions from left to right in u^-, i.e., starting from $u = -\infty$, assume f_{m+1} intersects U_m at some position ($u < 0$) from below, say with an edge of an upper enveloped curve f_k, where $k \le m$. As f_k must locate on or above U_m in u^- after (on the right of) the intersection point and f_{m+1} will intersect f_k at most one point in u^- (Lemma 1), f_k will form a shield to prevent f_{m+1} from intersecting U_m again in u^-. The proof is similar for U_m in u^+ and the sweep is from right to left, i.e., starting from $u = \infty$.

Theorem 4: *The transversal for a set of n convex polygons with a total of N vertices can be found in O(N + n log n) time.*

Proof: Let's consider the half-plane u^- in the dual plane only, and the half-plane u^+ can be handled similarly. As the vertices of the polygons are given in order, the upper and lower enveloped curves in the dual plane can be constructed in linear time, i.e., O(N) time. After that, it takes O(n *log* n) time to sort the enveloped curves at the negative infinity. Assume $\{P_1 \ldots P_m\}$ has a total of M vertices, the minima curve U_m will have at most O(M) edges (Lemma 2). Sweeping U_m and f_{m+1} (with k_{m+1} vertices) from left to right until they intersect or $u = 0$ takes at most $O(k_{m+1} + c)$ time, assuming there are c edges of U_{m+1} on the left of the intersection point or in the half-plane u^{-} if U_m and f_{m+1} do not intersect in u^{-}. By Lemma 3, those edges of U_m and f_{m+1} after (on the right of) the intersection point will not be processed, then U_{m+1} will have at most $M - c + k_{m+1}$ edges. Note that not all the edges in f_{m+1} are included in U_{m+1} and, for those edges in f_{m+1} which are included, they will be removed immediately if they are processed during the constructing of U_k for some $k > m+1$. As each edge in the enveloped curve is processed at most twice, by amortized analysis, the minima curve U_n can be constructed in $O(N)$ time.

Finally, sweeping the minima and maxima curves to see whether there exists a u such that $U_n(u) \geq V_n(u)$ takes at most $O(N)$ time. Therefore, the transversal problem can be solved in $O(N + n \log n)$ time.

4. ACKNOWLEDGEMENT

The authors are very grateful to an anonymous reviewer for a careful reading of the paper and pointing out an overlooked mistake in the paper.

5. REFERENCES

[1] M. Atallah and C. Bajaj, "Efficient Algorithms for Common Transversals", *Information Processing Letters*, Vol. 25, No. 2, (1987), pp. 87-91.

[2] D. Avis and M. Doskas, "Algorithms for High Dimensional StabbingProblem". *Theoretical Foundations of Computer Graphics and CAD*, (1990), pp.199-210.

[3] D. Avis, J. Robert and R. Wenger, "Lower bounds for line stabbing". *Information Processing Letters*, Vol. 33, (1989), pp. 59-62.

[4] K. Q. Brown, "Geometric transforms for fast geometric algorithms". Report CMU-CS-80-101, Department of Computer Science, Carnegie-Mellon University (1980).

[5] M. Dyer, "Linear Time Algorithms for Two- And Three-Variable Linear Programs". *SIAM Journal of Computer Science*, Vol. 13, No. 1, (1984), pp.31-45.

[6] H. Edelsbrunner, "Finding Transversals for Sets of Simple Geometric Figures". *Theoretical Computer Science*, Vol. 35, (1985), pp. 55-69.

[7] H. Edelsbrunner, H. A. Maurer, F. P. Preparata, A.L. Rosenberg, E. Weizland D. Wood, "Stabbing Line Segments". *BIT*, Vol. 22, (1982), pp. 274-281.

[8] N. Megiddo, "Linear-Time Algorithms for Linear Programs in R3 and Related Problems". *SIAM Journal of Computer Science*, Vol. 12, No. 4, (1983), pp. 759-776.

[9] N. Megiddo, "Linear Programming in linear time when the dimension is fixed". *Journal of the ACM*, Vol. 31, No.1, (1984) pp 114-127.