Joint Conference of 1996 International Computer Symposium
December 19~21, Kaohsiung, Taiwan, R.O.C.

An Algorithm for the Optimal Ranking Problem on Interval Graphs*

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Abstract

In this paper, we develop a sequential algorithm for the optimal ranking problem on interval graphs. Given a set of n intervals, our optimal ranking algorithm takes $O(\alpha n^3)$ time to find an optimal ranking on the corresponding interval graph, where α is the clique number of this interval graph.

1 Introduction

An *interval family* I is a set of intervals on the real line. Let V and E denote the vertex and edge set of a graph G, respectively. An undirected graph $G = (V, E)$ is called an *interval graph* if its vertices can be put into a one-to-one correspondence with an interval family I such that two vertices in V are connected by an edge of E if and only if their corresponding intervals have nonempty intersection [2]. Figure 1 shows an interval family and its corresponding interval graph. The interval family I is called an *interval model* of G. Conversely, G is referred to as the intersection graph of I, or the interval graph corresponding to I. Interval graphs have been extensively studied and used as models for many real world problems. For instance, they have applications in course scheduling, genetics, psychology, archaeology [2,7], job scheduling, and computer aided design [1,3].

A *ranking* of a graph G is a mapping, Γ, from the vertices of G to the natural numbers such that for every path between any two vertices u and v, $u \neq v$, with $\Gamma(u) = \Gamma(v)$, there exists at least one vertex w on that path with $\Gamma(w) > \Gamma(u) = \Gamma(v)$. The value $\Gamma(v)$ of a vertex v is the rank of vertex v [4]. A ranking is *optimal* if the largest rank assigned is the smallest among all rankings. And the *ranking number* $r(G)$ of a graph G is the largest rank assigned in any optimal ranking of G. The *optimal ranking problem* on a graph G is the problem of finding an optimal ranking on G. Figure 2 shows a ranking and an optimal ranking on a graph. Figures 3 and 4 show two different optimal rankings for the intervals in Figure 1.

The constraints for the ranking problem imply that two adjacent vertices cannot have the same rank. Hence this problem is a restriction of the vertices coloring problem.

There are many sequential and parallel algorithms provided for the optimal ranking problem on a tree [4,5,8,9,10]. Now, the best sequential $O(n)$ time algorithm are proposed by A.A. Schaffer [8]. And an optimal ranking on a cograph represented by its binary parse tree can be found in $O(log\ n)$ time with $O(n/log\ n)$ processors on an *EREW PRAM* model [6].

The organization of this paper is as follows. In Section 2 we give some definitions, properties and notations used in our method. In Section 3 we present our method. In Section 4 we give our optimal ranking algorithm to solve the optimal ranking problem on intervals. In Section 5 we show the correctness and complexity of ourmethod. In Section 6 we make some concluding remarks.

2 Definitions, Properties and Notations

In this section, we shall give some notations in an interval graph. Consider a family $I = \{I_i = [a_i, b_i] \mid a_i \leq b_i, 1 \leq i \leq n\}$ of intervals on the real line, where a_i is the left endpoint of interval i and b_i is its right endpoint. We number these intervals by increasing order of their right endpoints. In this paper, we say that interval x is greater than interval y if the right endpoint of interval x is greater than the right endpoint of interval y. In this case, we also say that interval y is smaller than interval x. Without loss of generality, we assume that these $2n$ endpoints all are distinct. Let $G = (V, E)$ be the interval graph of a family I of n intervals. We first introduce three operators.

$\cup$: the set union,

$\bigcup$: the interval union, and

$\bigcap$: the interval intersection.

For instance, if $I_1 = [1,4]$, $I_2 = [2,5]$ and $I_3 = [3,6]$, then $\{I_1\} \cup \{I_2\} \cup \{I_3\} = \{I_1, I_2, I_3\}$, $\{I_1\} \bigcup \{I_2\} \bigcup \{I_3\} = [1,6]$ and $\{I_1\} \bigcap \{I_2\} \bigcap \{I_3\} = [3,4]$.

We now introduce a cut operation which can partition a set of intervals I into three subsets. In the

*This work was supported by National Science Council, Republic of China.

following, we define some functions and give examples according to Figure 1.

Definition 1. An interval graph I can be cut at the right point b_i of some interval I_i, such that

$$I = L(I,i) \cup C(I,i) \cup R(I,i), \quad i = 1, \ldots, n-1$$

, where

$L(I,i)$ = the set of intervals $I_1, \ldots, I_i$ sorted by right endpoints.

$C(I,i)$ = the set of the intervals which contain b_i, excluding I_i,

= the set of the intervals which contain $b_i + \epsilon$, where ϵ is an infinitely small positive number.

$R(I,i)$ = the set of remaining intervals. □

Definition 2. Consider an interval graph which is cut at b_h such that $I = L(I,h) \cup C(I,h) \cup R(I,h)$. Then $R(I,h)$ can be partitioned into

$$R(I,h) = L(R(I,h),i) \cup C(R(I,h),i) \cup R(R(I,h),i)$$

, where $i \in \{R(I,h)\} \setminus \{$ the greatest interval in $R(I,h)\}$,

such that

$L(R(I,h),i)$ = the set of the intervals in $R(I,h)$ not greater than I_i.

$C(R(I,h),i)$ = the set of the intervals in $R(I,h)$ which contain b_i, excluding I_i

= the set of the intervals in $R(I,h)$ which contain $b_i + \epsilon$.

$R(R(I,h),i)$ = the set of the remaining intervals in $R(I,h)$.

$= R(I,i)$. □

If interval $I_h = 0$, then $R(I,h) = I$. We assume the right endpoint of I_0 is smaller than the left endpoints of any other intervals and $| \{I_0\} | = 0$.

Example 1. In Figure 1, the interval graph I can be partitioned into $I = L(I,1) \cup C(I,1) \cup R(I,1)$ such that $L(I,1) = \{I_1\}$, $C(I,1) = \{I_2, I_3, I_5\}$ and $R(I,1) = \{I_4, I_6, \ldots, I_{13}\}$. Then interval graph $R(I,1)$ can further be partitioned into $R(I,1) = L(R(I,1),6) \cup C(R(I,1),6) \cup R(R(I,1),6)$ such that $L(R(I,1),6) = \{I_4, I_6\}$, $C(R(I,1),6) = \{I_7, I_8, I_{12}\}$ and $R(R(I,1),6) = \{I_9, I_{10}, I_{11}, I_{13}\}$. □

Definition 3. $num(h,l,r)$
= the number of the intervals in $\{I_{l+1}, \ldots, I_r\}$ which contain b_l and have left endpoints greater than b_h
= the number of the intervals in $C(R(I,h),l)$, excluding the intervals which are greater than interval I_r. □

Definition 4. $CR(h,l,r,k)$
= the kth smallest interval in $\{I_{l+1}, \ldots, I_r\}$ which contains b_l and has left endpoint greater than b_h.
= the kth smallest interval in $C(R(I,h),l)$, excluding the intervals which are greater than I_r. □

Definition 5. $head(l,r)$
= the smallest interval in $R(I,l) \setminus \{$ intervals greater than $I_r\}$. □

Definition 6. $tail(l,r)$
= the greatest interval in $R(I,l) \setminus \{$ intervals greater than $I_r\}$. □

If $head(h,j) = i$ then interval I_i is called a head of interval I_h, where $j \geq i > h$.

Example 2. In Figure 1, if the interval graph I was cut at b_2, then $I = L(I,2) \cup C(I,2) \cup R(I,2)$ such that $L(I,2) = \{I_1, I_2\}$, $C(I,2) = \{I_3, I_5\}$ and $R(I,2) = \{I_4, I_6, \ldots, I_{13}\}$. We have $head(2,13) = 4$ and $tail(2,13) = 13$ because I_4 and I_{13} are the smallest and greatest intervals in $R(I,2)$, respectively. Now, if we cut $R(I,2)$ at b_4 then
$R(I,2) = L(R(I,2),4) \cup C(R(I,2),4) \cup R(R(I,2),4)$
$= \{I_4\} \cup \{I_6\} \cup \{I_7, \ldots, I_{13}\}$.

Clearly, $num(2,4,13) = | C(R(I,2),4) | = 1$ and $\{CR(2,4,13,k) \mid k = 1, \ldots, \alpha\} = C(R(I,2),4) = \{I_6\}$. Hence $CR(2,4,13,1)$ is I_6. From the above $head(4,13)$ is I_7 and $tail(4,13)$ is I_{13} because I_7 and I_{13} are the smallest and greatest intervals in $R(I,4) = R(R(I,2),4)$, respectively. □

Example 3. Consider another case. In Figure 1, if the interval graph I was cut at b_3, then $I = L(I,3) \cup C(I,3) \cup R(I,3)$ such that $L(I,3) = \{I_1, I_2, I_3\}$, $C(I,3) = \{I_5\}$ and $R(I,3) = \{I_4, I_6, \ldots, I_{13}\}$. We have $head(3,13) = 4$ and $tail(3,13) = 13$. Now, if we cut $R(I,3)$ at b_6 then
$R(I,3)$
$= L(R(I,3),6) \cup C(R(I,3),6) \cup R(R(I,3),6)$
$= \{I_4, I_6\} \cup \{I_7, I_8, I_{12}\} \cup \{I_9, I_{10}, I_{11}, I_{13}\}$.

Clearly,
$\{CR(3,6,9,k) \mid k = 1, \ldots, \alpha\}$
$= C(R(I,3),6) \setminus \{$ intervals greater than $I_9\}$
$= \{I_7, I_8, I_{12}\} \setminus \{I_{12}\} = \{I_7, I_8\}$, and
$num(3,6,9) = | \{I_7, I_8\} | = 2$.

So we have $CR(3,6,9,1)$ and $CR(3,6,9,2)$ are I_7 and I_8, respectively. From the above $head(6,9)$ is I_9 and $tail(6,9)$ is I_9 because I_9 is both the smallest and greatest interval in
$R(I,6) \setminus \{$ the intervals which are greater than $I_9\}$
$= \{I_9, I_{10}, I_{11}, I_{13}\} \setminus \{$ intervals greater than $I_9\}$
$= \{I_9\}$. □

Definition 7. $rank(h,i,j) =$
1. ranking number of $I_i, \ldots, I_j$ in $R(I,h)$, if $head(h,j) = i$ and $i \leq j$;
2. 0, if $i = j = 0$;
3. *(undefined), otherwise. □

Clearly, if $head(h,n) = i$ then $rank(h,i,j) = r(R(I,h) \setminus \{$ intervals greater than $I_j\})$. For instance, $rank(0,1,13) = r(R(I,0)) = r(I)$, $rank(5,7,13) = r(R(I,5))$ and $rank(3,4,11) = r(R(I,3) \setminus \{$intervals greater than $I_{11}\}) = r(R(I,3) \setminus \{I_{12}, I_{13}\})$, since $head(0,13) = 1$, $head(5,13) = 7$ and $head(3,11) = 4$.

Example 4. In Figure 4, $rank(0,1,13) = r(I) = 6$, $rank(8,10,13) = r(R(I,8)) = 3$, which is not 4 because I_{12} is not in $R(I,8) = \{I_{10}, I_{11}, I_{13}\}$. And $rank(5,8,13) = *(undefined)$ because I_8 is not the head of I_5. We make such an assignment because we will not use this rank when we compute the ranking number $rank(0,1,n)$ of the interval graph I. If we cut interval graph I at b_5 then we have to compute $rank(5,7,13)$ rather than $rank(5,8,13)$. Specifically, in Figure 4, we know that intervals

$I_1, I_4, I_7, I_9, I_{10}$ are heads of some intervals because intervals $I_1, I_4, I_7, I_9, I_{10}$ are the smallest intervals of $R(I,0) = \{I_1\}$, $R(I,1) = R(I,2) = R(I,3) = \{I_4, I_6, \ldots, I_{13}\}$, $R(I,4) = R(I,5) = \{I_7, \ldots, I_{13}\}$, $R(I,6) = R(I,7) = \{I_9, I_{10}, I_{11}, I_{13}\}$ and $R(I,8) = \{I_{10}, I_{11}, I_{13}\}$, correspondingly. That is, I_1 is the head of I_0; I_4 is the head of I_1, I_2 and I_3; I_7 is the head of I_4 and I_5; I_9 is the head of I_6 and I_7; and I_{10} is the head of I_8. □

The values of function $rank(h,i,j)$ are shown in Table 1.

Definition 8. $cut(h,i,j)$
= the smallest interval I_l such that $r(C(R(I,h),l)) + max\{r(L(R(I,h),l)), r(R(R(I,h),l) \setminus \{$ intervals greater than $I_j\})\}$ equals the ranking number of intervals $I_i, \ldots, I_j$ in $R(I,h)$, if $head(h,j) = i$. □

Example 5. In Figure 4, $cut(0,1,13)$ is I_3 because I_3 is the smallest interval satisfying $r(C(I,3)) + max\{r(L(I,3)), r(R(I,3))\} = 1 + max\{3,5\} = 6$ (the ranking number of the interval graph I), though intervals I_5, I_7 and I_8 also satisfy the above condition. And $cut(3,4,13)$ is I_8 because I_8 is the smallest interval satisfying $r(C(R(I,3),8)) + max\{r(L(R(I,3),8)), r(R(R(I,3),8))\} = 2 + max\{3,3\} = 5$ (the ranking number of intervals $I_4, \ldots, I_{13}$ in $R(I,3)$). The other $cut(h,i,j)$ functions used in our method are shown in Table 2. □

Definition 9.
$\Gamma(i)$ = the rank of interval I_i. □

A set of $\Gamma(i)$ functions leading to an optimal ranking is shown in Table 3.

3 Our method

In this section, we present our method for the optimal ranking problem on an interval graph. We use the example in Figure 1 to illustrate it. In our method, an interval graph is partitioned into $I = L(I,i) \cup C(I,i) \cup R(I,i)$, where $i = 1, \ldots, n-1$ and the ranking number of this interval graph is

$$min_{i=1,\ldots,n-1}\{r(C(I,i)) + max\{r(L(I,i)) + r(R(I,i))\}\}.$$

Our method consists of two phases. In phase 1, we find the ranking number of this interval graph. In phase 2, we give each interval a rank according to the result in phase 1. Figure 5 shows the process to obtain the ranking number by applying our method. In Figure 1, the ranking number of this interval graph is

$r(I)$
$= rank(0,1,13)$
$= min\{$
$r(C(I,1)) + max\{r(L(I,1)), r(R(I,1))\},$
$r(C(I,2)) + max\{r(L(I,2)), r(R(I,2))\},$
$r(C(I,3)) + max\{r(L(I,3)), r(R(I,3))\},$
$r(C(I,4)) + max\{r(L(I,4)), r(R(I,4))\},$
$r(C(I,5)) + max\{r(L(I,5)), r(R(I,5))\},$
$r(C(I,6)) + max\{r(L(I,6)), r(R(I,6))\},$
$r(C(I,7)) + max\{r(L(I,7)), r(R(I,7))\},$
$r(C(I,8)) + max\{r(L(I,8)), r(R(I,8))\},$
$r(C(I,9)) + max\{r(L(I,9)), r(R(I,9))\},$
$r(C(I,10)) + max\{r(L(I,10)), r(R(I,10))\},$
$r(C(I,11)) + max\{r(L(I,11)), r(R(I,11))\},$
$r(C(I,12)) + max\{r(L(I,12)), r(R(I,12))\}$ $\}$
$= min\{$
$r(\{I_2, I_3, I_5\}) + max\{r(\{I_1\}), r(\{I_4, I_6, \ldots, I_{13}\})\},$
$r(\{I_3, I_5\}) + max\{r(\{I_1, I_2\}), r(\{I_4, I_6, \ldots, I_{13}\})\},$
$r(\{I_5\}) + max\{r(\{I_1, I_2, I_3\}), r(\{I_4, I_6, \ldots, I_{13}\})\},$
$r(\{I_5, I_6\}) + max\{r(\{I_1, \ldots, I_4\}), r(\{I_7, \ldots, I_{13}\})\},$
$r(\{I_6\}) + max\{r(\{I_1, \ldots, I_5\}), r(\{I_7, \ldots, I_{13}\})\},$
$r(\{I_7, I_8, I_{12}\}) + max\{r(\{I_1, \ldots, I_6\}), r(\{I_9, I_{10}, I_{11}, I_{13}\})\},$
$r(\{I_8, I_{12}\}) + max\{r(\{I_1, \ldots, I_7\}), r(\{I_9, I_{10}, I_{11}, I_{13}\})\},$
$r(\{I_9, I_{12}\}) + max\{r(\{I_1, \ldots, I_8\}), r(\{I_{10}, I_{11}, I_{13}\})\},$
$r(\{I_{10}, \ldots, I_{13}\}) + max\{r(\{I_1, \ldots, I_9\}), r(\{\})\},$
$r(\{I_{11}, \ldots, I_{13}\}) + max\{r(\{I_1, \ldots, I_{10}\}), r(\{\})\},$
$r(\{I_{12}, I_{13}\}) + max\{r(\{I_1, \ldots, I_{11}\}), r(\{\})\},$
$r(\{I_{13}\}) + max\{r(\{I_1, \ldots, I_{12}\}), r(\{\})\}$ $\}$
$= min\{$
$3 + max\{1,5\}, 2 + max\{2,5\},$
$1 + max\{3,5\}, 2 + max\{3,5\},$
$1 + max\{4,5\}, 3 + max\{4,4\},$
$2 + max\{4,4\}, 2 + max\{4,3\},$
$4 + max\{4,0\}, 3 + max\{4,0\},$
$2 + max\{5,0\}, 1 + max\{5,0\}$ $\}$
$= min\{8,7,6,7,6,7,6,6,8,7,7,6\}$
$= 6.$

In the following, we will show more about how to get the ranking number of this interval graph I.

$r(I)$
$= r(\{I_1, \ldots, I_{13}\})$
$= r(C(I,3)) + max\{r(L(I,3)), r(R(I,3))\}$, where $r(C(I,3)) = r(\{I_5\}) = 1$.

$r(L(I,3))$
$= r(\{I_1, I_2, I_3\})$
$= r(C(L(I,3),1)) + max\{r(L(L(I,3),1)), r(R(L(I,3),1))\}$
$= r(\{I_2, I_3\}) + max\{r(\{I_1\}), r(\{\})\}$
$= 2 + max\{1,0\}$
$= 3.$

$r(R(I,3))$
$= r(\{I_4, I_6, \ldots, I_{13}\})$
$= r(C(R(I,3),8)) + max\{r(L(R(I,3),8)), r(R(R(I,3),8))\}$, where $r(C(R(I,3),8)) = r(\{I_9, I_{12}\}) = 2$.

$r(L(R(I,3),8))$
$= r(\{I_4, I_6, I_7, I_8\})$
$= r(C(L(R(I,3),8),4)) + max\{r(L(L(R(I,3),8),4)), r(R(L(R(I,3),8),4))\}$
$= r(\{I_6\}) + max\{r(\{I_4\}), r(\{I_7, I_8\})\}$
$= 1 + max\{1,2\}$
$= 3.$

$r(R(R(I,3),8))$
$= r(\{I_{10}, I_{11}, I_{13}\})$
$= r(C(R(R(I,3),8),10)) + max\{r(L(R(R(I,3),8),10)), r(R(R(R(I,3),8),10))\}$
$= r(\{I_{11}, I_{13}\}) + max\{r(\{I_{10}\}), r(\{\})\}$
$= 2 + max\{1,0\}$
$= 3.$

From the above, $r(R(I,3)) = 5$ and $r(L(I,3)) = 3$. Hence,

$$\begin{aligned} r(I) &= r(C(I,3)) + max\{r(L(I,3)), r(R(I,3))\} \\ &= 1 + max\{3,5\} \\ &= 6. \end{aligned}$$

Figure 5 shows the relevant cut operations to obtain $rank(0,1,13)$ in phase 1.

By now, we know the ranking number of this interval graph is $rank(0,1,13) = 6$ and $cut(0,1,13)$ is I_3. Because $CR(0,3,13,1)$ is I_5, we give rank 6 to I_5. By this way, we rank intervals $\{I_1,\ldots,I_3\}$ and intervals $\{I_4,I_6,\ldots,I_{13}\}$, recursively. For intervals $\{I_4,I_6,\ldots,I_{13}\}$, we know $cut(3,4,13)$ is I_8. Because $CR(3,4,13,1)$ is I_9, we give rank 5 to I_9. And $CR(3,4,13,2)$ is I_{12}, we give rank 4 to I_{12}, etc. In this way, we can get the rank $\Gamma(i)$ of I_i in Table 3. Figure 6 shows how to rank each interval in phase 2.

4 Algorithm

In this section, we present an algorithm for the optimal ranking problem on interval graphs. Our algorithm consists of two phases. In phase 1, we compute the ranking number of this interval graph. In phase 2, we give each interval a rank according to phase 1 by backtracking. Figure 4 shows the optimal ranking obtained by applying our algorithm. Our algorithm is Algorithm Optimal-ranking shown below.

Algorithm Optimal-ranking

Phase 1 : /* Find the ranking number. */
1. Sort n intervals by their right endpoints;
2. Initialize $num(h,i,j)$, $CR(h,i,j,k)$, $head(i,j)$, $tail(i,j)$, $cut(h,i,j)$ to nil, and set $rank(h,i,j) = *(undefined)$, $rank(h,0,0) = 0$, where $h,i,j = 0,\ldots,n$ and $k = 1,\ldots,\alpha$;
3. *for* $i = 1$ to $n-1$
    for $j = i+1$ to n
        find $num(0,i,j)$, $head(i,j)$, $tail(i,j)$,
            $CR(0,i,j,k)$, where $k = 1,\ldots,\alpha$;
    endfor;
  endfor;
4. *for* $h = n-1$ to 1
    if ($head(h,n) \neq nil$)
        find $num(h,i,j)$ and $CR(h,i,j,k)$,
            $k = 1,\ldots,\alpha$, where
            $i \in \{head(h,n),\ldots,tail(h,n)-1\}\backslash$
                $\{CR(0,h,n,k) \mid k = 1,\ldots,\alpha\}$,
            $j \in \{i+1,\ldots,n\}\backslash$
                $\{CR(0,h,n,k) \mid k = 1,\ldots,\alpha\}$;
    endif;
  endfor;
5. *for* $h = n-1$ to 0
    if ($head(h,n) \neq nil$)
        $i = head(h,n)$;
        $rank(h,i,i) = 1$;
        find $rank(h,i,j) = min_k\{num(h,k,j)+$
            $max\{rank(h,i,k),$
            $rank(k,head(k,j),tail(k,j))\}\}$, where
            $j \in \{i+1,\ldots,n\} \setminus \{CR(0,h,n,l) \mid$
                $l = 1,\ldots,\alpha\}$,
            $k \in \{i,\ldots,j-1\} \setminus \{CR(0,h,n,l) \mid$
                $l = 1,\ldots,\alpha\}$;
        Set $cut(h,i,j) = u$, where u is the smallest interval which satisfies $rank(h,i,j) = num(h,u,j) + max\{rank(h,i,u), rank(u,head(u,j),tail(u,j))\}$;
    endif;
  endfor;

Phase 2 : /* Assign ranks to the intervals. */
$ranking(0,1,n,rank(0,1,n))$;
func $ranking(h,i,j,p)$
{
  if ($j - i > 0$)
  {
    for $k = 1$ to $num(h,cut(h,i,j),j)$
        $\Gamma(CR(h,cut(h,i,j),j,k)) = p$;
        $p = p - 1$;
    endfor;
    $ranking(h,i,cut(h,i,j),p)$;
    $ranking(cut(h,i,j),head(cut(h,i,j),j),$
        $tail(cut(h,i,j),j),p)$;
  }
  elseif ($i \neq nil$)
    $\Gamma(CR(h,i,j,1)) = p$;
  endelseif;
  endif;
}

5 Correctness and Complexity

In this section, we give some notations and prove that our method is correct. The following notations will go through out this paper.

P = the ranking number of an interval graph.

For any optimal ranking of an interval graph, we define q, T and t_c as follows.

q = the greatest rank which appeared more than once in this interval graph.

T = the number of the ranks which appeared exactly once in this interval graph.
$= P - q$.

t_c = the number of the intervals which contain the real number c and have ranks appeared exactly once in this interval graph.

Figure 3 shows an interval graph with the ranking number $P = 6$. We will use this interval graph to explain the following Lemmas and Theorems.

Lemma 1. For any optimal ranking of a connected interval graph, the number of the intervals ranked the ranking number P must be 1.

Proof: If not, we assume $\Gamma(I_{i_1}) = \Gamma(I_{i_2}) = P$. Then we can find a path

$$I_{i_1} \to I_{j_1} \to I_{j_2} \to \ldots\ldots \to I_{j_r} \to I_{i_2}, \quad r \geq 0$$

such that $\Gamma(Ij_1), \Gamma(Ij_2), \ldots, \Gamma(I_{j_r}) \leq P$. This contradicts the definition of ranking. $\square$

Example 6. In Figure 3, the ranking number is 6 and only I_5 is ranked 6. $\square$

Lemma 2. If an interval graph has a ranking number P, for any optimal ranking, we can find the greatest rank q which appeared more than once. That is, ranks $q+1,\ldots,P$ appeared exactly once. Then there exists at least one real number $x = c$ such that every interval I_i containing c must be of rank greater than q.

Proof: If the interval graph is a clique, then $q = 0$ and ranks $1, \ldots, n$ appeared exactly once. Hence, for any real number

$$c \in \bigcap_{i \in \{1,\ldots,n\}} \{I_i\},$$

the intervals containing c must be of ranks greater than q.

If the interval graph is not a clique. By Lemma 1, there exists at least one rank (P, for example) appeared exactly once. We assume that we cannot find such a point c, that is, for any point

$$x \in \bigcup_{i \in \{q+1,\ldots,P\}} \{I_i\},$$

there exists an interval containing x whose rank is in $\{1, \ldots, q\}$. Then
$\bigcup_{i \in \{q+1,\ldots,P\}} \{intervals\ \ ranked\ \ i\}$
$\subseteq \bigcup_{i \in \{1,\ldots,q\}} \{intervals\ \ ranked\ \ i\}$
$\Longrightarrow I = \bigcup_{i \in \{1,\ldots,P\}} \{intervals\ \ ranked\ \ i\}$
$= \bigcup_{i \in \{1,\ldots,q\}} \{intervals\ \ ranked\ \ i\}$.

Because rank q appeared twice at least, there exists $\Gamma(I_{i_1}) = \Gamma(I_{i_2}) = q$. We can find a path

$$I_{i_1} \rightarrow I_{j_1} \rightarrow I_{j_2} \cdots\cdots \rightarrow I_{j_r} \rightarrow I_{i_2}, \quad r \geq 0$$

such that $\Gamma(I_{j_1}), \ldots, \Gamma(I_{j_r}) \leq q$. This contradicts the definition of ranking.

Therefore, we can find at least one real number c such that the intervals containing c must be of ranks greater than q. □

Example 7. In Figure 3, $P = 6$, $q = 3$, ranks $4, 5, 6$ appeared exactly once and we can find a real number $c \in (7, 8), (18, 19)$ such that every interval I_i containing c must be of rank greater than 3. □

Theorem 1. Consider the optimal ranking of an interval graph. We can find a real number c such that this interval graph can be partitioned into

$$I = W(I,c) \cup M(I,c) \cup E(I,c), where$$

$W(I,c)$ = the set of the intervals whose right endpoints are in the left of c,
$M(I,c)$ = the set of the intervals containing c whose ranks are greater than q,
$E(I,c)$ = the set of the intervals whose left endpoints are in the right of c,
$P = t_c + max\{r(W(I,c)), r(E(I,c))\}$, and
$t_c = \mid M(I,c) \mid = r(M(I,c))$.

Proof: By Lemma 2, we can find a real number c such that every interval I_i containing c must be of rank greater than q. Hence we have

$$I = W(I,c) \cup M(I,c) \cup E(I,c)$$

Now, we want to show that $max\{r(W(I,c)), r(E(I,c))\} = P - t_c$. We consider the following two cases :

case 1: $max\{r(W(I,c)), r(E(I,c))\} = s < P - t_c$. Then we can rank the intervals in $W(I,c)$ and $E(I,c)$ with the ranks in $\{1, \ldots, s\}$ and rank the intervals in $M(I,c)$ with the ranks in $\{s+1, \ldots, s+t_c\}$. Because any path from the intervals in $W(I,c)$ to the intervals in $E(I,c)$ must be via some intervals in $M(I,c)$. Hence the ranking number of this interval graph is $s + t_c < P - t_c + t_c = P$. This contradicts that P is the ranking number of this interval graph. So $max\{r(W(I,c)), r(E(I,c))\} \geq P - t_c$.

case 2: $max\{r(W(I,c)), r(E(I,c))\} = s > P - t_c$. Because the ranks of the intervals in $M(I,c)$ appear exactly once, the ranking number of this interval graph is $s + t_c > P - t_c + t_c = P$. This contradicts that P is the ranking number of this interval graph.

From the above, $P = t_c + (P - t_c) = t_c + max\{r(W(I,c)), r(E(I,c))\}$. This proof is complete. □

Example 8. In Figure 3, let $c \in (7, 8)$ or $c \in (18, 19)$ we can partition $I = W(I,c) \cup M(I,c) \cup E(I,c)$. There are two cases :

case 1: If $c \in (7, 8)$, then $W(I,c) = \{I_1, I_2, I_3\}$, $M(I,c) = \{I_5\}$ and $E(I,c) = \{I_4, I_6, \ldots, I_{13}\}$. We have $P = 6$, $t_c = \mid M(I,c) \mid = 1$, $r(W(I,c)) = 3$ and $r(E(I,c)) = 5$. Hence $P = t_c + max\{r(W(I,c)), r(E(I,c))\}$.

case 2: If $c \in (18, 19)$, then $W(I,c) = \{I_1, \ldots, I_8\}$, $M(I,c) = \{I_9, I_{12}\}$ and $E(I,c) = \{I_{10}, I_{11}, I_{13}\}$. We have $P = 6$, $t_c = \mid M(I,c) \mid = 2$, $r(W(I,c)) = 4$ and $r(E(I,c)) = 3$. Hence $P = t_c + max\{r(W(I,c)), r(E(I,c))\}$. □

Lemma 3. If an interval graph I with the ranking number P is partitioned into $I = W(I,c) \cup M(I,c) \cup E(I,c)$, where c is a real number and $\mid W(I,c) \mid = i$, such that $P = t_c + max\{r(W(I,c)), r(E(I,c))\}$. Then $r(C(I,i)) + max\{r(L(I,i)), r(R(I,i))\} = r(M(I,c)) + max\{r(W(I,c)), r(E(I,c))\}$.

proof: (I) We want to show that $r(C(I,i)) + max\{r(L(I,i)), r(R(I,i))\} \leq r(M(I,c)) + max\{r(W(I,c)), r(E(I,c))\}$.

If we choose $\mid L(I,i) \mid = \mid W(I,c) \mid = i$, then $L(I,i) = W(I,c)$. Because $W(I,c)$ is the set of the smallest i intervals sorted by the right endpoints. We know that if $\mid W(I,x) \mid = \mid W(I,y) \mid$ and $x < y$, then $\mid C(I,x) \mid \leq \mid C(I,y) \mid$. Because if $x < y$, there may be some interval with left endpoint z, $x < z < y$. In this case $\mid C(I,x) \mid < \mid C(I,y) \mid$. And it is impossible to have some interval with right endpoint z, $x < z < y$, because $\mid W(I,x) \mid = \mid W(I,y) \mid$. Now, we have $\mid C(I,i) \mid \leq \mid M(I,c) \mid$ because $\mid W(I,c) \mid = \mid L(I,i) \mid$ and $b_i < c$.

From the above, we have $\mid L(I,i) \mid = \mid W(I,c) \mid$ and $\mid C(I,i) \mid \leq \mid M(I,c) \mid$. Assume $\mid C(I,i) \mid = \mid M(I,c) \mid - k$, where k is a nonnegative integer. Hence $\mid R(I,i) \mid = \mid E(I,c) \mid + k$. So $r(C(I,i)) = r(M(I,c)) - k$ and $r(R(I,i)) \leq r(E(I,c)) + k$. We have

$$\begin{aligned} & r(C(I,i)) + max\{r(L(I,i)), r(R(I,i))\} \\ & = r(M(I,c)) - k + max\{r(W(I,c)), r(R(I,i))\} \\ & \leq r(M(I,c)) - k + max\{r(W(I,c)), r(E(I,c)) + k\} \\ & = r(M(I,c)) + max\{r(W(I,c)) - k, r(E(I,c))\} \\ & \leq r(M(I,c)) + max\{r(W(I,c)), r(E(I,c))\}. \end{aligned}$$

(II) We know that $r(M(I,c)) + max\{r(W(I,c)), r(E(I,c))\} = P$ is the ranking number. So, $r(M(I,c)) + max\{r(W(I,c)), r(E(I,c))\} \leq r(C(I,i)) + max\{r(L(I,i)), r(R(I,i))\}$.

By (I) and (II), this proof is complete. □

Example 9. Consider Figure 3. When $c \in (7,8)$, we have $i = 3$ and $r(C(I,3)) + max\{r(L(I,3)), r(R(I,3))\} = r(M(I,c)) + max\{r(W(I,c)), r(E(I,c))\} = 6$. And in this case, $C(I,3) = M(I,c)$, $L(I,3) = W(I,c)$ and $R(I,3) = E(I,c)$.

When $c \in (18,19)$, we have $i = 8$ and $r(C(I,8)) + max\{r(L(I,8)), r(R(I,8))\} = r(M(I,c)) + max\{r(W(I,c)), r(E(I,c))\} = 6$. And in this case, $C(I,8) = M(I,c)$, $L(I,8) = W(I,c)$ and $R(I,8) = E(I,c)$. □

Theorem 2. Our method provides a ranking number on interval graphs.

proof: According to our method, the interval graph I can be partitioned into $I = L(I,i) \cup C(I,i) \cup R(I,i)$, where $i = 1, \ldots, n-1$ and the ranking number of our method is

$ranking\_number$

$= min_{i=1,\ldots,n-1}\{r(C(I,i)) + max\{r(L(I,i)) + r(R(I,i))\}\}$

$= min_{i=1,\ldots,n-1}\{P_i\}$, where

$P_i = r(C(I,i)) + max\{r(L(I,i)) + r(R(I,i))\}$.

By Theorem 1, an interval graph I with the ranking number P can be partitioned into $I = W(I,c) \cup M(I,c) \cup E(I,c)$ such that $P = r(M(I,c)) + max\{r(W(I,c)), r(E(I,c))\}$. By Lemma 3, let $i = |W(I,c)|$, then $r(C(I,i)) + max\{r(L(I,i)), r(R(I,i))\} = r(M(I,c)) + max\{r(W(I,c)), r(E(I,c))\} = P$.

We want to show that $ranking\_number = P$. If not, there are two cases :

1. $ranking\_number > P$

This contradicts that $ranking\_number$ is the minimum of $\{P_1, \ldots, P_{n-1}\}$ and $P = P_i$, for some $i \in \{1, \ldots, n-1\}$.

2. $ranking\_number < P$

This contradicts that P is the ranking number of this interval graph.

This proof is complete. □

Theorem 3. Given an interval graph G with n unsorted intervals, our optimal ranking algorithm takes $O(\alpha n^3)$ time to solve the optimal ranking problem on interval graphs, where α is the clique number of this interval graph.

proof: The correctness has been shown in Theorem 2. We now analyze the time complexity of Algorithm Optimal-ranking. There are two phases in Algorithm Optimal-ranking. In Phase 1 : Step 1 takes $O(n \log n)$ time to sort n intervals by their right endpoints. Step 2 takes $O(\alpha n^3)$ time to initialize the values of $num(h,i,j)$, $CR(h,i,j,k)$, $head(i,j)$, $tail(i,j)$, $cut(h,i,j)$ and $rank(h,i,j)$. Step 3 takes $O(\alpha n^2)$ time to compute the values of $num(0,i,j)$, $head(i,j)$, $tail(i,j)$ and $CR(0,i,j,k)$. Step 4 takes $O(\alpha n^3)$ time to compute the values of $num(h,i,j)$ and $CR(h,i,j,k)$. Step 5 takes $O(\alpha n^3)$ to compute the values of $rank(h,i,j)$ and $cut(h,i,j)$. In Phase 2, we take $O(n)$ time to give each interval a rank. Hence Algorithm Optimal-ranking takes $O(\alpha n^3)$ time to solve the optimal ranking problem on interval graphs. □

6 Concluding Remarks

Some further research topics are as follows.

1. To reduce the time complexity of the optimal ranking problem on interval graphs.

2. To solve the optimal ranking problem on some other special graphs.

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h	i \ j	1	2	3	4	5	6	7	8	9	10	11	12	13
8	10	*	*	*	*	*	*	*	*	*	1	2	*	3
7	9	*	*	*	*	*	*	*	*	1	2	3	*	4
6	9	*	*	*	*	*	*	*	*	1	2	3	*	4
5	7	*	*	*	*	*	*	1	2	2	3	3	4	5
4	7	*	*	*	*	*	*	1	2	2	3	3	4	5
3	4	*	*	*	1	*	2	2	3	3	3	4	5	5
2	4	*	*	*	1	*	2	2	3	3	3	4	5	5
1	4	*	*	*	1	*	2	2	3	3	3	4	5	5
0	1	1	2	3	3	4	4	4	4	4	4	5	5	6

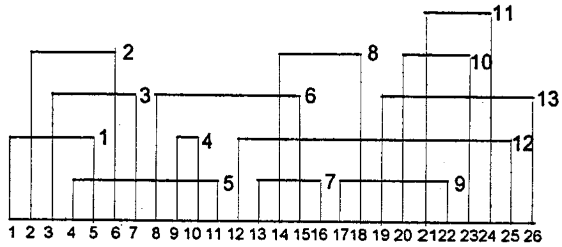
Table 1. The values of $rank(h,i,j)$, where $h,i,j = 0,\dots,n$.

$cut(0,1,13)$	3
$cut(0,1,3)$	1
$cut(3,4,13)$	8
$cut(3,4,8)$	4
$cut(4,7,8)$	7
$cut(8,10,13)$	10

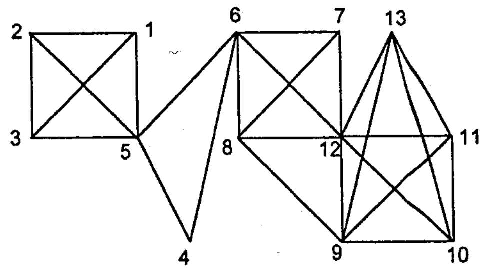
Table 2. The values of $cut(h,i,j)$ for $h,i,j = 0,\dots,n$.

i	1	2	3	4	5	6	7	8	9	10	11	12	13
$\Gamma(i)$	3	5	4	2	6	3	1	2	5	1	3	4	2

Table 3. A set of $\Gamma(i)$ functions leading to an optimal ranking.

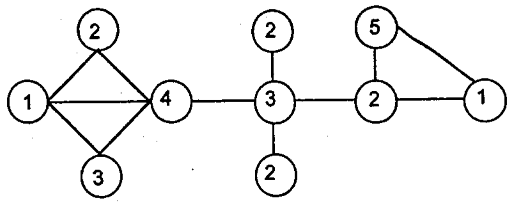


(a) A family of intervals sorted by their right endpoints.

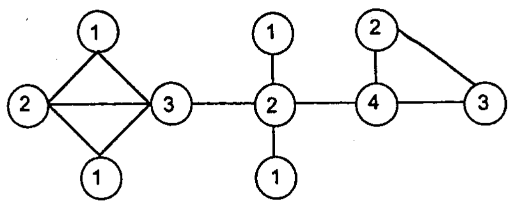


(b) The interval graph corresponding to the intervals in (a).

Figure 1. An interval family and its corresponding interval graph.



(a) A ranking on a graph.



(b) An optimal ranking on a graph.

Figure 2. A ranking (a) and an optimal ranking (b) on a graph.

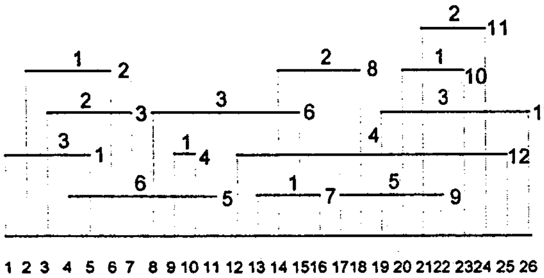


Figure 3. An optimal ranking of the interval grapgh in Figure 1.
The numbers above the intervals are their ranks.

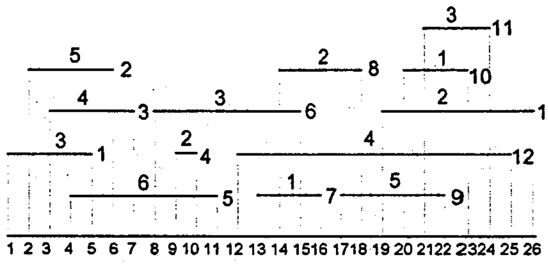


Figure 4. The optimal ranking obtained by applying our method.
The numbers above the intervals are their ranks.

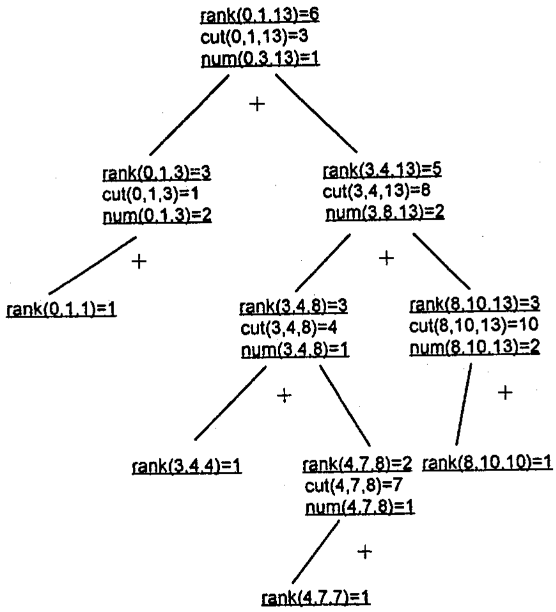


Figure 5. How to obtain rank(0,1,13) in phase 1.

The rank on a node can be computed once the ranks of its children are known. Thus a order to obtain the ranks are : rank(8,10,13)=3,rank(4,7,8)=2,rank(3,4,8)=3,rank(3,4,13) =5,rank(0,1,3)=3 and rank(0,1,13)=6.

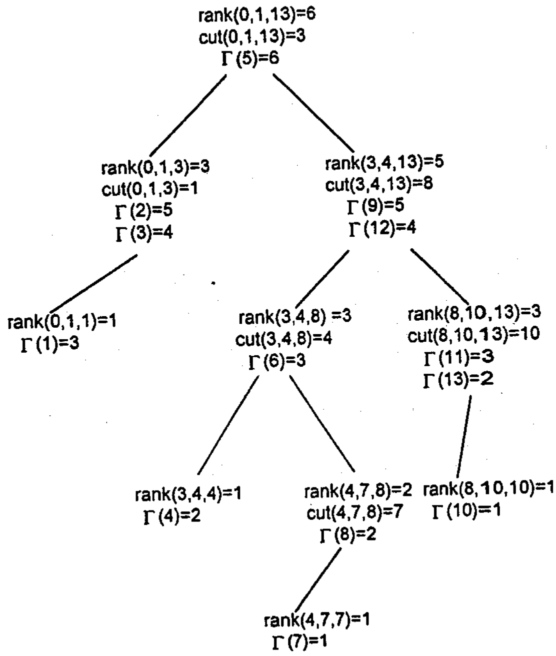


Figure 6. How to give each interval a rank in phase 2.