Fault-free Ring Embedding in Faulty Crossed Cubes *

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abstract

An n-dimensional crossed cube, CQ_n, is a variation from hypercube. In this paper, we prove that CQ_n is $(n-2)$-hamiltonian and $(n-3)$-hamiltonian connected. That is, a ring of length $2^n - f_v$ can be embedded in a faulty CQ_n with f_v faulty nodes and f_e faulty edges, where $f_v + f_e \leq n-2$ and $n \geq 3$. In other words, we show that the faulty CQ_n is still hamiltonian with $n-2$ faults. In addition, we also prove that there exists a hamiltonian path between any pair of vertices in a faulty CQ_n with $n-3$ faults. A recent result has shown that a ring of length $2^n - 2f_v$ can be embedded in a faulty Hypercube, if $f_v + f_e \leq n-1$ and $n \geq 4$, with a few additional constrains [10]. Our results, in comparison to Hypercube, show that longer rings can be embedded in CQ_n without additional constrains.

Keywords: *crossed cube, fault tolerant, hamiltonian, hamiltonian connected, hypercube.*

1. Introduction

The architecture of a local *network* is usually represented as a *graph*. A *ring* structure (*hamiltnian cycle*) is widely used in local networks, for its good properties such as low connectivity, simplicity, expandability, and easiness to implement. The embedding problem, which maps a source graph into a host graph, is an important topic of recent studies. The embedding of rings into various networks has been discussed. For example, a ring (*fault-tolerant* ring) can be embedded in faulty Stars [13], faulty arrangement graphs [6], double loop networks [11], de Bruijn networks [9], and faulty Hypercubes [8, 10, 12].

Hypercube is a popular network because of its attractive properties, including regularity, symmetry, small diameter, strong connectivity, recursive construction, partitionability, and relatively low link complexity [1, 8, 10, 12]. The *crossed cube*, a variation of Hypercube denoted as CQ_n, is derived by changing connections from Hypercube [2, 5]. The total number of *vertices* and *edges* in a crossed cube is the same as Hypercube. The crossed cubes have been studied recently, because they have several properties that are superior to Hypercubes [3, 4]. For example, the diameter of a crossed cube is nearly that of half of Hypercube. Hence, The number of average communication steps in the crossed cube is approximately half of Hypercube. As another example, a $(2^n - 1)$-node complete binary tree can be embedded into a 2^n-node crossed cube with dilation 1 [7]. However, a $(2^n - 1)$-node complete binary tree can only be embedded into a 2^n-node Hypercube with dilation 2 [14].

A recent result has shown that a ring of length $2^n - 2f_v$ can be embedded in a faulty Hypercube, if $f_v + f_e \leq n-1$ and $n \geq 4$, with a few additional constrains [10]. In this paper, we will show that a ring of length $2^n - f_v$ can be embedded in a faulty CQ_n with f_v faulty nodes and f_e faulty edges, where $f_v + f_e \leq n-2$ and $n \geq 3$. All of the fault-free vertices can be included in the ring in the faulty CQ_n. In other words, we will show that the faulty CQ_n is still hamiltonian with $n-2$ faults. This result is optimal, since there is no hamiltonian cycle in a *regular graph* with degree n, which can hold over $n-2$ faults. We will also prove that there exists a *hamiltonian path* between any pair of vertices in a faulty CQ_n with $n-3$ faults. This result is also optimal. The reason is as follows: Assume that there are $n-2$ faults in a crossed cube CQ_n. It is possible that there exists a vertex v with degree 2 in this faulty CQ_n. Let x and y be the two vertices adjacent to v, then x and y can not be the end points of any hamiltonian path, since such a path must traverse both v and other vertices, which is not possible.

The rest of this paper is organized as follows. Section two explains notations and the basic properties of crossed cubes. The main theorem is proved in section three. The conclusion is given in section four.

2. Notations and basic properties

Let $G = (V, E)$ be an undirected graph. We refer to

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[5] for our fundamental graph terminology, when using undirected graph to model *interconnection networks*. Given a graph, the vertex set and the edge set of G are denoted by $V(G) = V$ and $E(G) = E$, respectively. A *path*, $P(v_0, v_t) = \langle v_0, v_1, \ldots, v_t \rangle$, is a sequence of nodes such that two consecutive nodes are *adjacent*. A path $\langle v_0, v_1, \ldots, v_t \rangle$ may contain other subpath, denoted as $\langle v_0, v_1, \ldots, v_i,\ P(v_i, v_j),\ v_j, v_{j+1},\ \ldots, v_t \rangle$, where $P(v_i, v_j) = \langle v_i, v_{i+1}, \ldots, v_{j-1}, v_j \rangle$. A path that contains every vertex of G exactly once is called a hamiltonian path of G. A path $\langle v_0, v_1, \ldots, v_t, v_0 \rangle$ is called a *cycle*, if v_0 is adjacent to v_t and $t \geq 3$. A cycle which visits each vertex in G exactly once is called a hamiltonian cycle. A graph that contains a hamiltonian cycle is called a *hamiltonian graph* (or simply *hamiltonian*). A graph G is called *hamiltonian connected* if there exists a hamiltonian path between any two vertices of G.

A graph $G - F$ denotes the subgraph of G with node faults and/or edge faults, i.e., a faulty network, where $F \subset V(G) \bigcup E(G)$. Let k be a positive integer. A graph G is *k-hamiltonian* if $G - F$ is hamiltonian for any F with $|F| \leq k$. That is, a ring can still be embedded into a faulty network $G - F$. Similarily, a graph G is *k-hamiltonian connected* if $G - F$ is hamiltonian connected for any F with $|F| \leq k$.

We now introduce some definitions.

Definition 1 *Two two-digit binary strings $x = x_1x_0$ and $y = y_1y_0$ are pair related, denoted as $x \sim y$, if and only if $(x, y) \in (00, 00), (10, 10), (01, 11), (11, 01)$.*

Definition 2 *An n-dimension crossed cube CQ_n is a graph $CQ_n = (V, E)$ that is recursively constructed as follows: CQ_1 is a complete graph with two vertices labeled by 0 and 1, respectively. CQ_n consists of two identical $(n-1)$-dimension crossed cubes CQ^0_{n-1} and CQ^1_{n-1}. The vertex $u = 0u_{n-2} \ldots u_0 \in V(CQ^0_{n-1})$ and vertex $v = 1v_{n-2} \ldots v_0 \in V(CQ^1_{n-1})$ are adjacent in CQ_n if and only if*

(1) $u_{n-2} = v_{n-2}$ if n is even, and

(2) for $0 \leq i < \lfloor \frac{n-1}{2} \rfloor$, $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$.

We denote CQ^0_{n-1} and CQ^1_{n-1} as the subcrossed cubes of CQ_n. In addition, we define the edge set $E_c = \{(u_0, u_1) \mid (u_0, u_1) \in E, u_0 \in CQ^0_{n-1}$ and $u_1 \in CQ^1_{n-1}\}$ of CQ_n as the set of crossing edges of CQ_n. For any edge $e = (u_0, u_1) \in E_c$, the vertices u_0 and u_1 are called crossing nodes of each other. Clearly, there are 2^{n-1} crossing edges and 2^{n-1} pairs of crossing nodes in CQ_n.

Examples of CQ_3 and CQ_4 are shown in Fig. 1. From above two definitions, we know that every vertex in CQ_n with a leading bit 0 has exactly one neighbor with a leading bit 1 and vice versa. It is obvious that, CQ_n is connected, regular graph of degree n with 2^n vertices.

In Hypercube, there is a simple rule that an edge (u, v) exists, if and only if u differs from v in exactly one bit. However, the rule for a crossed cube is a little more complicated. According to definition 2, for all $n \geq 1$, the edge $(u = u_{n-1}u_{n-2} \ldots u_l u_{l-1} \ldots u_0, v = v_{n-1}v_{n-2} \ldots v_l v_{l-1} \ldots v_0)$ exists in CQ_n if and only if there exists an l with

(1) $u_{n-1}u_{n-2} \ldots u_l = v_{n-1}v_{n-2} \ldots v_l$,

(2) $u_{l-1} = \overline{v_{l-1}}$,

(3) $u_{l-2} = v_{l-2}$ if l is even, and

(4) for $0 \leq i < \lfloor \frac{l-1}{2} \rfloor$, $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$.

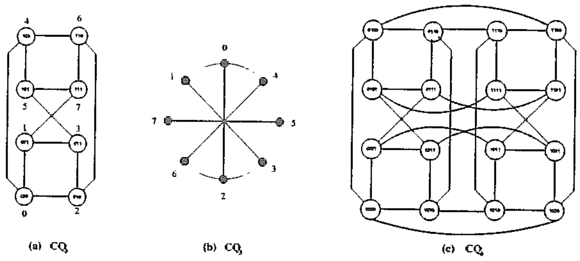


Figure 1: CQ_n for n=3,4.

If both of the rules (1) and (2) hold, we say that u and v have a *leftmost differing bit at position $l - 1$*. We refer to rulers (3) and (4) as the pairing rules. If l is odd, the rule (3) is not used.

3. Hamiltoian Cycles in Crossed Cubes

We will prove that a ring of length $2^n - f_v$ can be embedded in CQ_n with f_v faulty nodes and f_e faulty edges, where $f_v + f_e \leq n - 2$. That is, we will prove that CQ_n is $(n-2)$-hamiltonian, for $n \geq 3$. Moreover, we will show that CQ_n is $(n-3)$-hamiltonian connected, for $n \geq 3$. For simplifying our proof, we are not distinct from the f_e and f_v, and use $|F| = f_e + f_v$. Our proof is by induction on n, and the outline of our proof is as follows: First, for the induction base, we prove that CQ_3 is 1-hamiltonian and hamiltonian connected, and CQ_4 is 2-hamiltonian and 1-hamiltonian connected. Next, assuming CQ_k is $(k-2)$-hamiltonian and $(k-3)$-hamiltonian connected, for $4 \leq k \leq n$, we will show that CQ_{n+1} is $(n-1)$-hamiltonian and $(n-2)$-hamiltonian connected. To start our induction, let us look at the CQ_3 and CQ_4.

Recall that Fig. (1-a) and (1-b) are two isomorphic CQ_3's, where the vertices of Fig. (1-a) are labeled with binary numbers and the vertices of Fig. (1-b) are labeled with decimal numbers.

Lemma 1 *CQ_3 is 1-hamiltonian and hamiltonian connected.*

Proof: (1). It is easy to show that CQ_3 is 1-hamiltonian. For example, if the node 1 is faulty, $\langle 0, 2, 3, 5, 7, 6, 4, 0 \rangle$ is a fault-free hamiltonian cycle. Since Fig. (1-b) is a node symmetric graph, it is obvious that there are fault-free Hamiltonian cycles for any node fault.

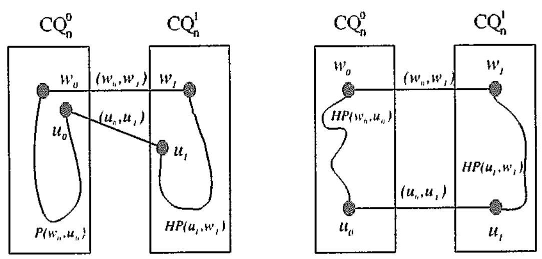
For instance, if the edge (0,4) is faulty, $\langle 0,1,7,6,4,5,3,2,0\rangle$ is a fault-free hamiltonian cycle and if the edge (0,2) is faulty, $\langle 0,1,7,6,2,3,5,4,0\rangle$ is another fault-free hamiltonian cycle. Since Fig. 1-(b) is a symmetric graph, there are fault-free Hamiltonian cycles for any edge fault.

(2). We want to show that CQ_3 is hamiltonian connected too. From Fig. (1-b), we can find that $\langle 0,2,6,7,1,3,5,4\rangle$, $\langle 0,4,6,2,3,1,7,5\rangle$, $\langle 0,2,6,4,5,7,1,3\rangle$ and $\langle 0,4,5,3,1,7,6,2\rangle$ are hamiltonian paths between the node 0 and 4, 0 and 5, 0 and 3, and 0 and 2, respectively. Since Fig. (1-b) is a node symmetric graph, there are Hamiltonian paths between the node 0 and 6, 0 and 7, 0 and 1. Therefore, CQ_3 is hamiltonian connected. □

For simplicity, we assume that CQ_4 is 2-hamiltonian and 1-hamiltonian connected. Then, we will denoted $P(w,u)$ as a path $\langle w,\dots u\rangle$ between w and u, and $HP(w,u)$ as a hamiltonian path between w and u in faulty or fault-free CQ_n. In addition, we use the notation HC_0 to denoted a hamiltonian cycle in faulty or fault-free CQ_n^0 and HC_1 as a hamiltonian cycle in faulty or fault-free CQ_n^1.

Lemma 2 *If CQ_n is $(n-2)$-hamiltonian and $(n-3)$-hamiltonian connected, CQ_{n+1} is $(n-1)$-hamiltonian, where $n \geq 4$.*

Proof: Let E_c be the set of crossing edges, i.e., $E_c = \{(u_0,u_1) \mid (u_0,u_1) \in E, u_0 \in CQ_n^0 \text{ and } u_1 \in CQ_n^1\}$. Let F be a faulty set of CQ_{n+1} with $F_0 = F \cap CQ_n^0$, $F_1 = F \cap CQ_n^1$, and $F_c = F \cap E_c$, and let $f_0 = |F_0|$, $f_1 = |F_1|$, $f_c = |F_c|$. We will show that CQ_{n+1} is $(n-1)$-hamiltonian, in the following three cases: (1) The faults are scattered (at least two of f_0, f_1, and f_c are greater then zero). (2) All of the faults are located in the same CQ_n (either $f_0 > 0, f_1 = 0, f_c = 0$ or $f_0 = 0, f_1 > 0, f_c = 0$). (3) All of the faults are located in E_c ($f_0 = 0, f_1 = 0, f_c > 0$).



(a) case 1 (b) case 2

Figure 2: Illustration for Lemma 2.

Case 1: The faults are scattered in CQ_n^0, CQ_n^1, and E_c, shown in Fig. (2-a): Without loss of generality, we assume that $f_0 \geq f_1$. If at least two of f_0, f_1 and f_c are greater than zero, then $f_1 \leq f_0 \leq n-2$. We want to prove that $f_1 \leq n-3$. f_1 is strictly either less than $n-2$ or equal to $n-2$. Suppose $f_1 = n-2$ then $f_0 = 1$. Since $f_0 \geq f_1$, then $1 \geq n-2$ and $3 \geq n$, contradicting the fact that $n \geq 4$. Thus, $f_1 \leq n-3$ and $f_1 + f_c \leq n-2$, where $n \geq 4$. Since CQ_n^0 is $(n-2)$-hamiltonian and $f_0 \leq n-2$, there exists a hamiltonian cycle, HC_0, with at least $2^n - (n-2)$ edges. We now show that there exists an edge $(w_0,u_0) \in HC_0$ such that the crossing nodes w_1 and u_1 of w_0 and u_0 respectively are both fault-free and the crossing edges (w_0,w_1) and (u_0,u_1) are also fault-free. Since $|HC_0| \geq 2^n-(n-2)$, we have at least $2^n-(n-2)$ choices. If none of the edges of HC_0 meets the requirements of (w_0,u_0), then there are at least $\lceil \frac{2^n-(n-2)}{2} \rceil$ faults in F_1 and F_c. (because a single fault in either F_1 or F_c eliminates at most 2 edges of HC_0), contradicting the fact that $f_1 + f_c \leq n-2$, for $n \geq 4$. Therefore, we can find such an edge (w_0,u_0) and then $HC_0 = \langle w_0, P(w_0,u_0), u_0, w_0\rangle$. Because CQ_n^1 is $(n-3)$-hamiltonian connected and $f_1 \leq n-3$, there exists a hamiltonian path between u_1 and w_1, i.e., $HP(u_1,w_1)$. Hence, $\langle w_0, P(w_0,u_0), u_0, u_1, HP(u_1,w_1), w_1, w_0\rangle$ is a fault-free hamiltonian cycle between x and y in CQ_{n+1}.

Case 2: All of the faults are located in the same CQ_n^i, for $i = 0,1$, shown in Fig. (2-b): Without loss of generality, we assume that all of the faults are located in CQ_n^0 and $f_0 = n-1$. Since CQ_n^0 is $(n-2)$-hamiltonian, there exists two vertices w_0 and u_0, such that there is a hamiltonian path between w_0 and u_0, say $HP(w_0,u_0)$. Let w_1 be the crossing node of w_0 and u_1 be the crossing node of u_0. We know that $w_1, u_1, (w_0,w_1)$ and (u_0,u_1) are all fault-free, because there are no faults in either E_c or CQ_n^1. Furthermore, since CQ_n^1 is hamiltonian connected, there exists a hamiltonian path between u_1 and w_1, i.e., $HP(u_1,w_1)$. Hence, $\langle w_0, HP(w_0,u_0), u_0, u_1, HP(u_1,w_1), w_1, w_0\rangle$ is a fault-free hamiltonian cycle in CQ_{n+1}.

Case 3: All of the faults $\in E_c$. Because there are 2^n crossing edges in CQ_{n+1}, there are at least $(2^n-(n-1)) \geq 2$ fault-free crossing edges, where $n \geq 4$. We can choose two fault-free crossing edges (w_0,w_1) and (u_0,u_1). Since both CQ_n^0 and CQ_n^1 are $(n-3)$-hamiltonian connected, there exist $HP(w_0,u_0)$ in CQ_n^0 and $HP(u_1,w_1)$ in CQ_n^1. Hence, $\langle w_0, HP(w_0,u_0), u_0, u_1, HP(u_1,w_1), w_1, w_0\rangle$ is a fault-free hamiltonian cycle in CQ_{n+1}. This completes the proof of the lemma. □

We need the following auxiliary lemma in Lemma 4. One may skip the poof temporarily, and come back for it afterwards.

Lemma 3 *Assume that CQ_{n-1} is hamiltonian connected. In a fault-free CQ_n with 4 distinct vertices w,u,x, and y, if $w \in CQ_{n-1}^0$ and $u \in CQ_{n-1}^1$, there exist two disjoint paths between w and x, and u and y or between w and y, and u and x. Moreover, these two disjoint paths traverse all vertices of CQ_n.*

Proof: Case (a): Since CQ_{n-1}^0 is hamiltonian connected, there exists a hamiltonian path between

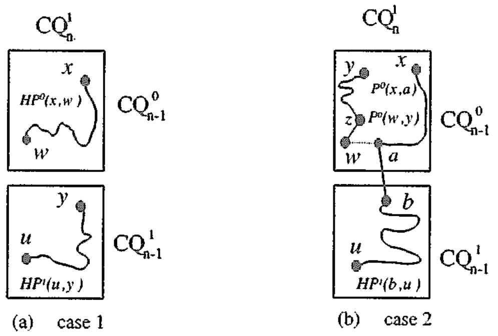


Figure 3: Illustration for Lemma 3.

x and w, i.e., $HP(x,w)$. Similarly, there exists a hamiltonian path $HP(u,y)$ between u and y. Hence, $\langle x, HP(x,w), w\rangle$ and $\langle u, HP(u,y), y\rangle$ are the two disjoint paths, shown in Fig. (3-a).

Case (b): x and y are in the same sub-crossed cube CQ_{n-1} of CQ_n, shown in Fig. (3-b): Without loss of generality, we assume that both x and $y \in CQ^0_{n-1}$. Since CQ^0_{n-1} is hamiltonian connected, there exists a hamiltonian path $HP(x,y)$ between x and y. Let a and z be the two neighboring nodes of w on $HP(x,y)$. Then, $HP(x,y) = \langle x, P(x,a), a, w, z, P(z,y), y\rangle$. And let b be the crossing node of a. Assume $b \neq u$. Then, there exists a hamiltonian path between b and u because CQ_{n-1} is hamiltonian connected. Hence, $\langle x, P(x,a), a, b, HP(b,u), u\rangle$ and $\langle w, z, P(z,y), y\rangle$ are the two disjoint paths. In the case that $b = u$, we can simply replace a with z, the similar argument as above still holds. □

Lemma 4 *If CQ_{n-1} is $(n-3)$-hamiltonian and $(n-4)$-hamiltonian connected and CQ_n is $(n-2)$-hamiltonian and $(n-3)$-hamiltonian connected, then CQ_{n+1} is $(n-2)$-hamiltonian connected, where $n \geq 4$.*

Proof: We will show that there exists a hamiltonian path between every pair of vertices x and y in CQ_{n+1} with $|F| \leq n-2$. There are three cases: (1) The faults are scattered (at least two of f_0, f_1, and f_c are greater than zero). (2) All of the faults are located in the same CQ_n (either $f_0 > 0, f_1 = 0, f_c = 0$ or $f_0 = 0, f_1 > 0, f_c = 0$). (3) All of the faults are located in E_c ($f_0 = 0, f_1 = 0$, and $f_c > 0$).

Case 1: The faults are scattered in CQ^0_n, CQ^1_n, and E_c. Without loss of generality, we assume that $f_0 \geq f_1$. Because any two of f_0, f_1 and f_c are greater than zero and $f_1 \leq f_0 \leq n-3$, thus $f_1 \leq n-3$ and $f_1 + f_c \leq n-3$, where $n \geq 4$.

There are three subcases: (1.1) $x \in CQ^0_n$ and $y \in CQ^1_n$. (1.2) both x and $y \in CQ^0_n$. (1.3) both x and $y \in CQ^1_n$.

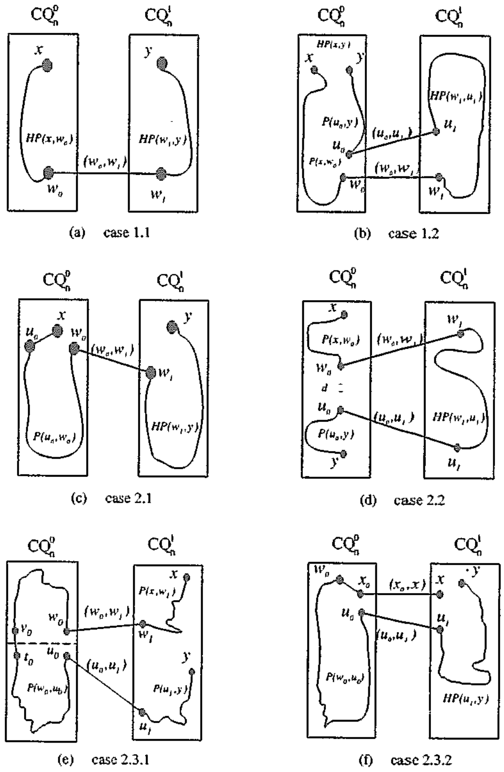


Figure 4: Illustration for Lemma 4.

(1.1) x and y are in different CQ^i_n, for $i = 0, 1$, shown in Fig. (4-a): Because there are 2^n crossing edges in CQ_{n+1}, we have at least $(2^n - (n-2)) \geq 3$ fault-free crossing edges, for $n \geq 4$. Let (w_0, w_1) be one of the fault-free crossing edges, $w_0 \neq x$, and $w_1 \neq y$. Since CQ^0_n is $(n-3)$-hamiltonian connected and $f_0 \leq n-3$, there exist a fault-free hamiltonian path $HP(x, w_0)$ in CQ^0_n. Similarly, since CQ^1_n is $(n-3)$-hamiltonian connected and $f_1 \leq n-3$, there also exist a fault-free hamiltonian path $HP(w_1, y)$ in CQ^1_n. Hence, $\langle x, HP(x,w_0), w_0, w_1, HP(w_1,y), y\rangle$ is a fault-free hamiltonian path between x and y in CQ_{n+1}.

(1.2) Both x and y are in the same CQ^i_n, for $i = 0, 1$, shown in Fig. (4-b): Without loss of generality, we assume that $i = 0$. Since CQ^0_n is $(n-3)$-hamiltonian connected and $f_0 \leq n-3$, there exists a hamiltonian path $HP(x,y)$ between x and y. Since $|HP(x,y)| \geq 2^n - (n-3)$, and we have at least $2^n - (n-3)$ choices, where $n \geq 4$. We can find an edge (w_0, u_0) on the path $HP(x,y)$ such that the crossing node w_1 and u_1 of w_0 and u_0 respectively are both fault-free and the crossing edges (w_0, w_1) and (u_0, u_1) are also fault-free. Then, $HP(x,y) = \langle x, P(x,w_0), w_0, u_0, P(u_0,y), y\rangle$. Since CQ^1_n is $(n-3)$-hamiltonian connected and $f_1 \leq n-3$, there exists a hamiltonian path between w_1 and u_1, i.e., $HP(w_1, u_1)$. Hence, $(HP(x,y) \cup \{(w_0,w_1),(u_0,u_1)\} \cup HP(w_1,u_1)) - \{(w_0,u_0)\}$ is a fault-free hamiltonian path in CQ_{n+1}.

(1.3) Both x and y are $\in CQ_n^1$: This case is symmetric to the case (1.2).

Case 2: All of the faults are in the same CQ_n.

Assume all of faults are located in CQ_n^0. There are three subcases: (2.1) $x \in CQ_n^0$ and $y \in CQ_n^1$ (2.2) both x, and $y \in CQ_n^0$ (2.3) both x and $y \in CQ_n^1$.

(2.1) x and y are in different CQ_n^i, for $i = 0, 1$, shown in Fig. (4-c): Without loss of generality, we assume that $i = 0$ and $f_0 = n - 2$. Since CQ_n^0 is $(n-2)$-hamiltonian, there exists a hamiltonian cycle with vertices u_0 and w_0 adjacent to x, i.e., $HC_0 = \langle x, u_0, P(u_0, w_0), w_0, x\rangle$. Let w_1 be the crossing node of w_0 and u_1 be the crossing node of u_0. We know that (w_0, w_1) and (u_0, u_1) are fault-free, because there are no faults in E_c. Since CQ_n^1 is hamiltonian connected, there exists a hamiltonian path between w_1 and y, i.e., $HP(w_1, y)$. Hence, if $w_1 \neq y$, $\langle x, u_0, P(u_0, w_0), w_0, w_1, HP(w_1, y), y\rangle$ is a fault-free hamiltonian path between x and y in CQ_{n+1}. Otherwise, $\langle x, w_0, P(w_0, u_0), u_0, u_1, HP(u_1, y), y\rangle$ is a fault-free hamiltonian path between x and y in CQ_{n+1}.

(2.2) Both x, and y are $\in CQ_n^0$, shown in Fig. (4-d): Let d be a fault of F. Since CQ_n^0 is $(n-3)$-hamiltonian connected, $CQ_n^0 - (F - \{d\})$ contains a hamiltonian path between x and y, i.e., $HP(x, y)$. Thus, $CQ_n^0 - F$ contains two node-disjoint paths $P(x, w_0)$ and $P(u_0, y)$, where $P(x, w_0) \cup P(u_0, y) = HP(x, y) - \{d\}$. Because CQ_n is $(n-3)$-hamiltonian connected and $n - 3 \geq 0$, there exists a hamiltonian path between w_1 and u_1, i.e., $HP(w_1, u_1)$. Hence, $\langle x$, $P(x, w_0)$, w_0, w_1, $HP(w_1, u_1)$, u_1, u_0, $P(u_0, y), y\rangle$ is a hamiltonian path between x and y in CQ_{n+1}.

(2.3) Both x and $y \in CQ_n^1$.

There are another two subcases in this case. Let $x_0 \in CQ_n^0$ be the crossing node of x and $y_0 \in CQ_n^0$ be the crossing node of y.

(2.3.1) Both x_0 and y_0 are faulty, shown in Fig. (4-e): Since CQ_n^0 is $(n-2)$-hamiltonian and $f_0 = n - 2$, there exists a hamiltonian cycle HC_0. Since HC_0 is a hamiltonian cycle, there are at least two edges crossing the two sub-crossed cubes of CQ_n^0. Let one of the edges be (w_0, u_0) and w_1 be the crossing node of w_0, and u_1 be the crossing node of u_0. Since w_0 and u_0 belong to different sub-crossed cube of CQ_n^0, w_1 and u_1 must also belong to different sub-crossed cube of CQ_n^1. In addition, x_0 and y_0 are both faulty, therefore w_1 and u_1 can't be either x or y. Since CQ_n^1 is fault-free, by Lemma 3, we have four district vertices u_1, w_1, x, y, therefore there are two disjoint paths, which traverse through all vertices of CQ_n^1, say, $\langle x, P(x, w_1), w_1\rangle$ and $\langle u_1$, $P(u_1, y)$, $y\rangle$. Hence, $\langle x, P(x, w_1), w_1, w_0$, $P(w_0, u_0)$, $u_0, u_1, P(u_1, y), y\rangle$ is a fault-free hamiltonian path between x and y in CQ_{n+1}.

(2.3.2) At least one of x_0 or y_0 is fault-free, shown in Fig. (4-f): Assume x_0 is fault-free. Since CQ_n^0 is $(n-2)$-hamiltonian and $f_0 = n - 2$, there exists a hamiltonian cycle containing vertex x_0, i.e., $HC_0 = \langle x_0, w_0$, $P(w_0, u_0)$, $u_0, x_0\rangle$. Let u_1 be crossing node of u_0 and $u_1 \neq y$ (if $u_1 = y$, we can simply use w_0 to replace u_0). Since CQ_n^1 is $(n-3)$-hamiltonian connected and $n - 3 \geq 1$, there exists $HP(u_1, y)$ in $CQ_n^1 - \{x\}$. Hence, $\langle x, x_0, w_0$, $P(w_0, u_0)$, u_0, u_1, $HP(u_1, y), y\rangle$ is a fault-free hamiltonian path between x and y in CQ_{n+1}.

Case 3: All of the faults $\in E_c$.

There are also three subcases: (3.1) $x \in CQ_n^0$ and $y \in CQ_n^1$ (3.2) both x and $y \in CQ_n^0$ (3.3) both x and $y \in CQ_n^1$.

(3.1) $x \in CQ_n^0$ and $y \in CQ_n^1$. The conditions of this case are in fact similar to the case (1.1). The same arguments used in case (1.1) can also be applied here to obtain a fault-free hamiltonian path between x and y.

(3.2) Both x and $y \in CQ_n^0$. The conditions of this case are in fact similar to the case (1.2). We can find an edge (w_0, u_0) from CQ_n^0 and fault-free vertices and edges $w_1, u_1, (w_0, w_1)$, and (u_0, u_1) from CQ_n^1 and E_c with the fact that $(2^n - 2(n-2)) \geq 2$. Therefore, a similar hamiltonian path between x and y as in the case (1.2) can be found.

(3.3) This case is symmetric to the case (3.2). This completes the proof of the lemma. □

Now we are ready to prove our main theorem.

Theorem 1 *CQ_n is (n-2)-hamiltonian and (n-3)-hamiltonian connected, for $n \geq 3$.*

Proof: By Lemma 1, CQ_3 is 1-hamiltonian and hamiltonian connected in addition to assume that CQ_4 is 2-hamiltonian and 1-hamiltonian connected. Then, by Lemma 2 and 4, CQ_n is (n-2)-hamiltonian and (n-3)-hamiltonian connected. Therefore, by induction, this theorem is true. □

4. Conclusions

This paper focuses on the study of a faulty crossed n-cube, $CQ_n-(f_v+f_e)$, which contains f_v faulty nodes and f_e faulty edges. We prove that a ring of length $2^n - f_v$ can be embedded in a faulty CQ_n with $f_v+f_e \leq n-2$. That is, all of the fault-free vertices can be included in the ring in the faulty CQ_n. This result is optimal, since there is no hamiltonian cycle in a regular graph with degree n, which can hold over $n-2$ faults. We have also proved another optimal result that there exists a hamiltonian path between any pair of vertices in a faulty CQ_n with $n-3$ faults.

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