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The Gray-Scale Soft Morphological Filters and the Idempotency

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Abstract

Gray-scale soft mathematical morphology is the natural extension of binary soft mathematical morphology which has been proved to be less sensitive to additive noise and to small variations. But gray-scale soft morphological operations are difficult to implement in real time. In this paper, a superposition property called threshold decomposition and another property called stacking are introduced and have been found to apply successfully to gray-scale soft morphology operations. These properties allow the gray-scale signals and structuring elements to be decomposed into their binary sets, respectively operated by only logic gates in new VLSI architecture, and then these binary results are combined to produce the desired output as the time-consuming gray-scale processing. The theorems developed may significantly improve speed as well as give new theoretical insight into the operations. At last, the new class of idempotent soft morphological filters are presented. Theorems and proofs are provided.

1. Introduction

Mathematical morphology [2,3,8,9] which is based on set theory, provides an algebraic approach to manifest structuring shapes on binary or gray-scale images. Morphological filters [12] constitute a highly nonlinear system. Another popular family of nonlinear filters is the order statistic filters which are based on order statistics [1] and have been applied to signal detection and image enhancement. Maragos and Schafer [6] extended the theory of median, order statistic and stack filters to mathematical morphology. They have shown that the order statistic filters can be used in both function- and set-processing and can commute with thresholding.

Soft morphological filters [4,5,11,13] are the combination of the order statistic filters and morphological filters. The primary difference from standard morphological filters is that the maximum and minimum iterations are replaced by the more general weighted order statistics and the *soft* boundary is added to the structuring element. Koskinen *et al.* [4, 5] have shown that soft morphological operations are less sensitive to additive noise and to small variations in object shape and preserve most of the desirable properties of standard morphological operations.

Let f be a function defined on m-dimensional discrete Euclidean space Z^m. Let W be a window of a finite subset in Z^m and $N = Card(W)$, the *cardinality* of W. The k-th order-statistics (OS^k) of a function $f(x)$ with respect to the window W is a function whose value at location x is obtained by sorting in descendent order the N values of $f(x)$ inside the window W whose origin is shifted to location x and picking up the k-th number from the sorted list, where k ranges from 1 to N. Let W_x denote the translation of the origin in W to the location x. The k-th OS is represented by

$$OS^k(f,W)(x) = k\text{-th largest of } \{f(a) | a \in W_x\}.$$

In this paper, the new definitions of gray-scale soft morphological operations are introduced and their properties with threshold superposition are presented. The gray-scale soft morphological operations are proved to be capable of commuting with thresholding, and the property of threshold superposition allows fast implementation of function-processing soft morphological operations by using only logic gates. The implementation and the analysis of function-processing soft morphological operations can be interpreted by focusing only on the case of sets that are much easier to deal with since set-processing soft morphological operations only involve in counting the number of pixels instead of sorting the values.

The paper is organized as follows. In section 2 & 3 the definitions of binary and gray-scale soft morphological operations are given. In section 4 the threshold decomposition of gray-scale soft morphological dilation is discussed. In section 5 the threshold decomposition of gray-scale soft morphological erosion is discussed. In section 6 the idempotent soft morphological filters are presented. Finally, conclusions are made.

2. Soft Morphological Operations of Function-by-Set and Set-by-Set

The soft morphological operations adopt the concept of order statistics to replace the maximum and minimum in standard morphological operations. The basic idea of soft morphological operations is that the structuring element B is split into two subsets: the core set A and the soft boundary set $B \backslash A$, where "\" denotes the set difference. Soft morphological dilation (erosion) of a function with respect to a finite set is defined where output value at location x is obtained by sorting in descendent (ascendant) order of the $Card(B \backslash A) + k \times Card(A)$ values of the input image including the pixels inside $B \backslash A$ and repetition k times of pixels inside A, and then selecting the k-th order from the sorted list. Let $\{k * f(a)\}$ denote the repetition k times of $f(a)$ that means $\{k * f(a)\} = \{f(a), f(a), ..., f(a)\}$ (k times). The definitions of soft morphological operations of function-by-set are given as follows, where f is a gray-scale image and A and B is a flat structuring elements.

Definition 1: The soft morphological dilation of f by $[B, A, k]$ is defined as

$$(f \oplus [B,A,k](x) = k\text{-th largest of } \{k * f(a) \mid a \in A_x\} \cup \{f(b) \mid b \in (B \backslash A)_x\}).$$

Definition 2: The soft morphological erosion of f by $[B, A, k]$ is defined as

$$(f \odot [B,A,k])(x) = k\text{-th smallest of } \{k * f(a) \mid a \in A_x\} \cup \{f(b) \mid b \in (B \backslash A)_x\}).$$

Definition 3: The soft morphological closing of f by $[B, A, k]$ is defined as

$$f \bullet [B,A,k] = (f \oplus [B,A,k]) \odot [B,A,k] .$$

Definition 4: The soft morphological opening of f by $[B, A, k]$ is defined as

$$f \odot [B,A,k] = (f \odot [B,A,k]) \oplus [B,A,k] .$$

The soft morphological operations of set-by-set are defined as follows.

Definition 5: The soft morphological dilation of X by $[B, A, k]$ is defined as

$$X \oplus [B,A,k] = \{x \mid k \times Card(X \cap A_x) + Card(X \cap (B \backslash A)_x) \geqq k\} .$$

Definition 6: The soft morphological erosion of X by $[B, A, k]$ is defined as

$$X \oplus [B,A,k] = \{x \mid k \times Card(X \cap A_x) + Card(X \cap (B \backslash A)_x) \geqq N + (k-1) \times n-k+1\}.$$

Referring to eq. (6), if any one pixel within A_x is one or the number of ones within $(B \backslash A)_x$ is greater or equal to k then the output is one. Similarly in eq. (7), if any one pixel within A_x is one and the number of ones within $(B \setminus A)_x$ is greater or equal to $N-n-k+1$ (i.e. the number of zeros is greater than or equal to k), then the output is one.

Proposition 1: The soft morphological operations are increasing.

Proposition 2: The soft morphological operations are translation-invariant. That is for any soft morphological operation Ψ with respect to any $z \in Z^m$, we have

$$\Psi_z(f) = \Psi(f_z) .$$

The proof of translation invariance property is straightforward and therefore skipped. The cross-section $X_t(f)$ of f at level t is the set obtained by thresholding f at level t:

$$X_t(f) = \{x \mid f(x) \geqq t\}, \text{ where } -\infty < t < \infty .$$

Theorem 1: The soft morphological operations of function-by-set commute with thresholding. That is for any t, we have

$$X_t(f \oplus [B,A,K]) = X_t(f) \oplus [B,A,K].$$

The essence of Theorem 1 is that the soft morphological operations of function-by-set followed by thresholding at level t is equivalent to thresholding the function at level t followed by soft morphological filtering of set-by-set of the resultant cross section. Both ways yield the same result. The implementation and analysis of soft morphological operations of function-by-set can be achieved using the set-by-set operations which are much easier to deal with because their definitions involves in only counting the number of pixels instead of sorting numbers. Consider the thresholded binary images

$$f_a(x) = \begin{cases} 1 & \text{if } f(x) \geq a \\ 0 & \text{otherwise} \end{cases},$$

where $0 \leq a \leq L$ and L is the largest value in f. Most of the cases we use gray levels from 0 to 255, i.e. L=255. It is simple to show that f can be reconstructed from its thresholded binary images. That is

$$f(x) = \sum_{a=1}^{L} f_a(x) = \max\{a \mid f_a(x) = 1\}.$$

A transformation Ψ is called to posses *threshold-linear superposition* [5] if it satisfies

$$\Psi(f) = \sum_{a=1}^{L} \Psi(f_a).$$

Such a transformation Ψ can be realized by decomposing f into all its binary thresholded images f_a's. Next, we will show that the soft morphological operations of function-by-set obey the threshold-linear superposition.

Theorem 2: The soft morphological operations of function-by-set obey the threshold-linear superposition. That is

$$f \oplus [B, A, k] = \sum_{a=1}^{L} \left[f_a \oplus [B, A, k]\right].$$

3. Soft Morphological Operations of Function-by-Function

The standard binary morphological operations of dilation, erosion, opening, and closing are all naturally extended to gray-scale by the use of a *min* or *max* operation [8,14]. The definitions of soft morphological operations of function-by-function are defined as follows, where f is the gray-scale image and α and β are the gray-scale structuring element sets. Let the symmetric function $g^s(x)$ with respect to the origin be given by

$$g^s(x) = g(-x).$$

If F is the domain of f, B is the domain of β, and A is the domain of α. The definitions of soft morphological operations of function-by- function are given as follows.

Definition 7: The soft morphological dilation of f by $[\beta, \alpha, k]$ is

$$f \oplus [\beta, \alpha, k](z) = \text{k - th largest of}$$
$$\left(\{k * (f(y) + \alpha(z - y))\} \cup \{f(b) + \beta(z - b)\}\right),$$

where $z\text{-}y \in A$ and $z\text{-}b \in B \backslash A$.

Definition 8: The soft morphological erosion of f by $[\beta, \alpha, k]$ is

$$f \odot [\beta, \alpha, k](z) = \text{k - th smallest of}$$
$$\left(\{k * (f(y) - \alpha(z + y))\} \cup \{f(b) - \beta(z + b)\}\right),$$

where $z\text{-}y \in A$ and $z\text{-}b \in B \backslash A$.

Definition 9: The soft morphological closing of f by $[\beta, \alpha, k]$ is

$$f \bullet [\beta, \alpha, k](x) = (f \oplus [\beta, \alpha, k]) \odot [\beta, \alpha, k](x).$$

Definition 10: The soft morphological opening of f by $[\beta, \alpha, k]$ is

$$f \circ [\beta, \alpha, k](x) = (f \odot [\beta, \alpha, k]) \oplus [\beta, \alpha, k](x).$$

One of the most important links between sets and functions is the *umbra* which was introduced by Sternberg [12]. The *umbra* of a function f is defined as

$$U[f] = \{(x, y) \mid 0 \le y \le f(x)\},$$

where only positive value is considered. The *top surface* of a set A is defined as

$$T[A](x) = \max\{y \mid (x, y) \in A\}..$$

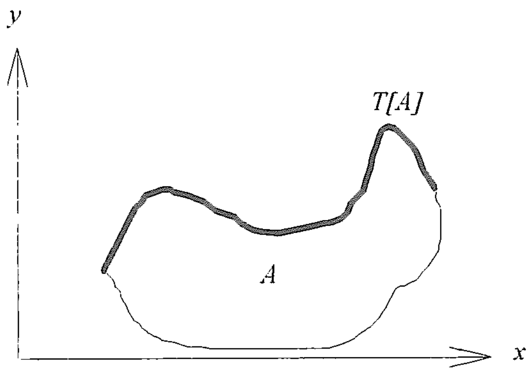

Fig. 1. The top surface T[A] of a set A.

After the operations of taking a top surface of a set and an umbra of a surface are defined, we have the following theorems.

Theorem 3: The soft morphological dilation of function-by-function is the top surface of the soft morphological dilation of their umbras. That is

$$f \oplus [\beta, \alpha, k] = T\{U[f] \oplus [U^s[\beta], U^s[\alpha], k]\}.$$

Theorem 4: The soft morphological erosion of function-by-function is the top surface of the soft erosion of their umbras. That is

$$f \odot [\beta, \alpha, k] = T\{U[f] \oplus [U^s[\beta], U^s[\alpha], k]\}.$$

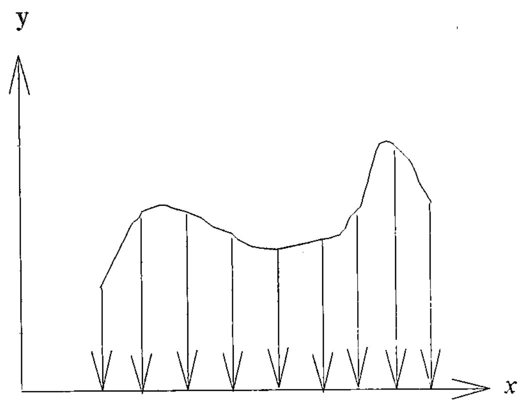

Fig. 2. The umbra U[T[A]] of a function T[A].

4. Threshold Decomposition Algorithm for Soft Morphological Dilation of Function-by-Function

In this section, we will show that soft morphological dilation of function-by-function commute with thresholding and obey threshold superposition. Let a *slice* and *slice complement* [10] of a function *f*, denoted by *S*[*f*] and *S'*[*f*] respectively, be defined as

Definition 11: The slice of a function *f* is

$$S[f_i](x,y) = \begin{cases} 1 & \textbf{if } y = i \textbf{ and } f(x) \ge y \\ 0 & \textbf{otherwise} \end{cases}.$$

Definition 12: The slice complement of a function *f* is

$$S[f_i](x,y) = \begin{cases} 0 & \textbf{if } y = i \textbf{ and } f(x) < y \\ 1 & \textbf{otherwise} \end{cases}.$$

Let *I* and *J* be the maximal gray-scale values of *f* and β respectively, i.e., $I = \max\{a | f_a = 1\}$ and $J = \max\{a | \alpha_a = 1 \textbf{ or } \beta_a = 1\}$. The umbra of a function *f* can be decomposed into *I* slice complements. That is

$$U[f] = \bigcup_{i=0}^{I} S[f_i].$$

From Theorem 3, we have

$$f \oplus [\beta,\alpha,k] = T\{U[f] \oplus [U[\beta^S], U[\alpha^S], k]\}$$

$$= T\left\{\bigcup_{i=0}^{I} S[f_i] \oplus [U[\beta^S], U[\alpha^S], k]\right\}$$

$$= T\left\{\bigcup_{i=0}^{I}\bigcup_{j=0}^{J}\left[S[f_i] \oplus \left[S[\beta_j^S], S[\alpha_j^S], k\right]\right]\right\}$$

$$= \max T\left\{\bigcup_{j=0}^{J}\left[S[f_i] \oplus \left[S[\beta_j^S], S[\alpha_j^S], k\right]\right]\right\}.$$

Based upon Definition 11, each slice of the input signal and structure element sets consists of only one non-zero row. It is possible to perform the binary soft morphological dilation in one less dimension. The top surface operation in the above equation can be replaced by a summation of all reduced-dimensionality stacking signals.

$$f \oplus [\beta,\alpha,k] =$$

$$\max \left\{ \begin{matrix} \left[f_0 \oplus \left[\beta_j^S, \alpha_j^S, k\right]\right] - 1, \\ \sum_{j=0}^{J} \sum_{j=0}^{J} \left[f_1 \oplus \left[\beta_j^S, \alpha_j^S, k\right]\right], \\ \sum_{j=0}^{J} \left[f_2 \oplus \left[\beta_j^S, \alpha_j^S, k\right]\right] + 1, \cdots \end{matrix} \right\}$$

Let the input signal and structuring element sets be non-negative. The first term in the above equation is a constant. That is

$$\sum_{j=0}^{J} \left[f_0 \oplus \left[\beta_j^S, \alpha_j^S, k\right]\right] - 1 \doteq J.$$

The following equation can be derived.

$$f \oplus [\beta,\alpha,k] = J +$$

$$\max \left\{ f_1 \oplus \left[\beta_J^S, \alpha_J^S, k\right], \sum_{j=J-1}^{J} \left[f_2 \oplus \left[\beta_J^S, \alpha_J^S, k\right]\right], \cdots \right\}.$$

Based upon the above equation, the threshold decomposition algorithm for soft morphological dilation of function-by-function is described as follows.

1. Determine the highest gray-scale values of the input signal and the structuring element sets, denoted by *I* and *J* respectively.

2. Assume $I \ge J$. Decompose *f* into *I* binary images $\{f_a | 1 < a \le I\}$ and decompose β into *J*+1 binary structuring element sets $\{\beta_a | 0 \le a \le J\}$.

3. Compute the binary results Y_{ia}, that is
$$Y_{ia} = f_i \oplus [B, A, k]$$
$$\textbf{for} \begin{cases} J - i + 1 \le a \le J & \textbf{if } 1 \le i \le J \\ 0 \le a \le J & \textbf{if } J < i \le I. \end{cases}$$

4. Sum up all the stacking binary results to obtain Y_i.
$$Y_i = \begin{cases} \sum_{a=J-i+1}^{J} Y_{ia} & \textbf{if } 1 \le i \le J \\ \sum_{a=0}^{J} Y_{ia} & \textbf{if } J < i \le I. \end{cases}$$

5. Compute the gray-scale result (Y) by selecting the maximum value of Y_i at each position x and then add J to every position.

$$Y(x) = \max Y_i(x) + J.$$

An example to illustrate the threshold decomposition of soft morphological dilation of function-by-function shown in Fig. 3.

5. Threshold Decomposition of Soft Morphological Erosion of Function-by-Function

In this section, we will show that soft morphological dilation of function-by-function commute with thresholding and obey threshold superposition. Let a *slice* and *slice complement* [10] of a function f, denoted by $S[f]$ and $S'[f]$ respectively, be defined as

Definition 11: The slice of a function f is

$$S[f_i](x,y) = \begin{cases} 1 & \text{if } y = i \text{ and } f(x) \geq y \\ 0 & \text{otherwise} \end{cases}.$$

Definition 12: The slice complement of a function f is

$$S[f_i](x,y) = \begin{cases} 0 & \text{if } y = i \text{ and } f(x) < y \\ 1 & \text{otherwise} \end{cases}.$$

The umbra of a function f can be decomposed into I slice complements. That is

$$U[f] = \bigcap_{i=0}^{I} S'[f_i].$$

From Theorem 4, we have

$$f \odot [\beta, \alpha, k] = T\{U[f] \odot [U^S[\beta], U^S[\alpha], k]\}$$
$$= T\left\{\bigcap_{i=0}^{I} S'[f_i] - \left[\bigcup_{j=0}^{J} S[\beta_j^S], \bigcup_{j=0}^{J} S[\alpha_j^S], k\right]\right\}.$$

The following equation can be derived.

$$f \odot [\beta, \alpha, k] = \{[f_1 \odot [\beta_J^S, \alpha_J^S, k]] + \cdots + [f_1 \odot [\beta_J^S, \alpha_J^S, k] \cap f_0 \odot [\beta_{J-1}^S, \alpha_{J-1}^S, k]] + \cdots +$$
$$[f_J \odot [\beta_J^S, \alpha_J^S, k] \cap f_{J-1} \odot [\beta_{J-1}^S, \alpha_{J-1}^S, k] \cap \cdots \cap f_1 \odot [\beta_1^S, \alpha_1^S, k]] + \cdots +$$
$$[f_I \odot [\beta_J^S, \alpha_J^S, k] \cap f_{I-1} \odot [\beta_{J-1}^S, \alpha_{J-1}^S, k] \cap \cdots \cap f_{I-J} \odot [\beta_0^S, \alpha_0^S, k]]\} - J.$$

An example to illustrate the threshold decomposition of soft morphological erosion of function-by-function shown in Fig. 4.

6. Idempotent Soft Morphological Filters

An idempotent filter maps a class of input signals into an associated set of root sequences. Each root signals are invariant to additional filter passes. Since the nature of the soft morphological operations and their non-linearity, idempotency usually does not exist in such operations at the first stage. In this section, a new class of idempotent soft morphological filters are presented.

Proposition 3: The soft morphological closing fills up the valley whose area is less than or equal to *Card(A)*, where k=*Card(B\A)*. When the area of valley is equal to *Card(A)*, there must be no other valley in *B\A*.

When K = *Card(B\A)*, the valley with size *Card(A)* will not be filled fully. But the other valley within *B\A* will be filled up after soft morphological dilation. After one iteration of soft morphological closing, the valley within *B\A* will be disappeared and the valley with size *Card(A)* remain the same. Apply soft morphological dilation again, the valley with size *Card(A)* will be filled up. Thus, the idempotency will be held after 2 iterations when this special case occurs.

Theorem 5: For any f and finite sets B and A, the soft morphological closing is idempotent if *k=Card(B\A)*.

Theorem 6: For any f and finite sets B and A, the soft morphological opening is idempotent if *k=Card(B\A)*.

Theorem can be derived similarly. Examples are given to show soft morphological closing is idempotent when *k=Card(B\A)*. Core sets and structuring elements set are shown in Fig. 5. Input image is shown in Fig. 6. Results shown in Figs. 7 and 8 are the idempotent cases, i.e. if the operation is applied again, the same results are obtained. Fig. 9 shows that when $k \neq Card(B \setminus A)$ it is not idempotent and results will be changed by iterations.

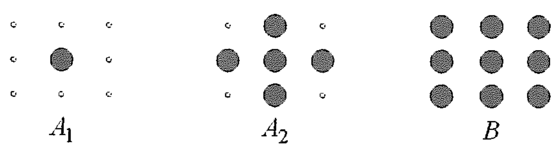

Fig. 5. Two core sets A_1, A_2 and the structuring element **B**.

0	0	0	0	0	0	0
0	1	2	2	1	1	0
0	2	4	5	7	9	0
0	2	3	3	2	2	0
0	1	2	1	2	1	0
0	4	4	5	5	4	0
0	0	0	0	0	0	0

Fig. 6 The original image *f*.

0	0	0	0	0	0	0
0	1	2	2	1	1	0
0	2	4	5	7	7	0
0	2	3	3	2	2	0
0	1	2	2	2	1	0
0	4	4	5	5	4	0
0	0	0	0	0	0	0

Fig. 7 The result of $\Psi(f)$ **with** A_1, B, **and** $k = 8$.

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	4	5	7	7	0
0	0	3	4	5	2	0
0	0	4	4	4	1	0
0	0	4	5	5	4	0
0	0	0	0	0	0	0

Fig. 8 The result of $\Psi(f)$ **with** A_2, B, **and** $k = 4$.

0	0	0	0	0	0	0
0	0	1	2	1	0	0
0	1	2	3	2	1	0
0	2	3	3	3	2	0
0	2	3	3	3	2	0
0	0	2	3	2	0	0
0	0	0	0	0	0	0

(a)

0	0	0	0	0	0	0
0	0	0	1	0	0	0
0	0	2	2	2	0	0
0	1	2	3	2	1	0
0	0	2	3	2	0	0
0	0	0	2	0	0	0
0	0	0	0	0	0	0

(b)

Fig. 9 The result of (a) $\Psi[f]$ and (b) $\Psi[\Psi[f]]$ with A_1, B, **and** $k = 4$.

7. Conclusions

The concept of threshold decomposition is ideally suited to VLSI implementation of soft morphological operations. We have presented the threshold superposition of gray-scale soft morphological erosion into binary soft morphological erosion, which allows real-time implementation by using only logic gates. This is a general architecture decomposition provides not only the significant improvements in computation but also the inside aspects of gray-scale soft morphology. The application of gray-scale morphology to vision problems is still in active ongoing research area. A new class of idempotent soft morphological filters is presented which will lead us to reach the root signals without iterations.

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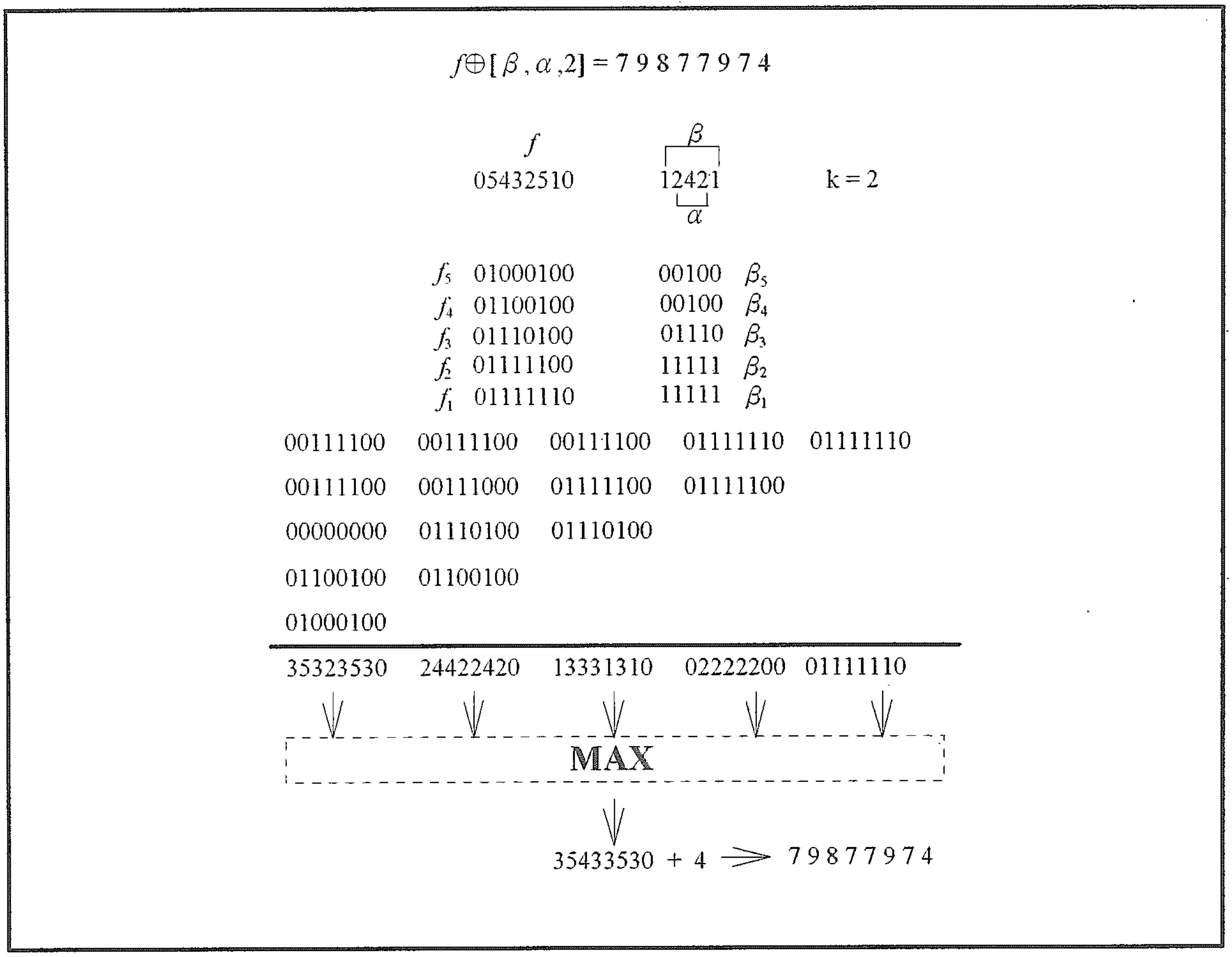

Fig. 3. Threshold decomposition of soft morphological dilation of function-by-function.

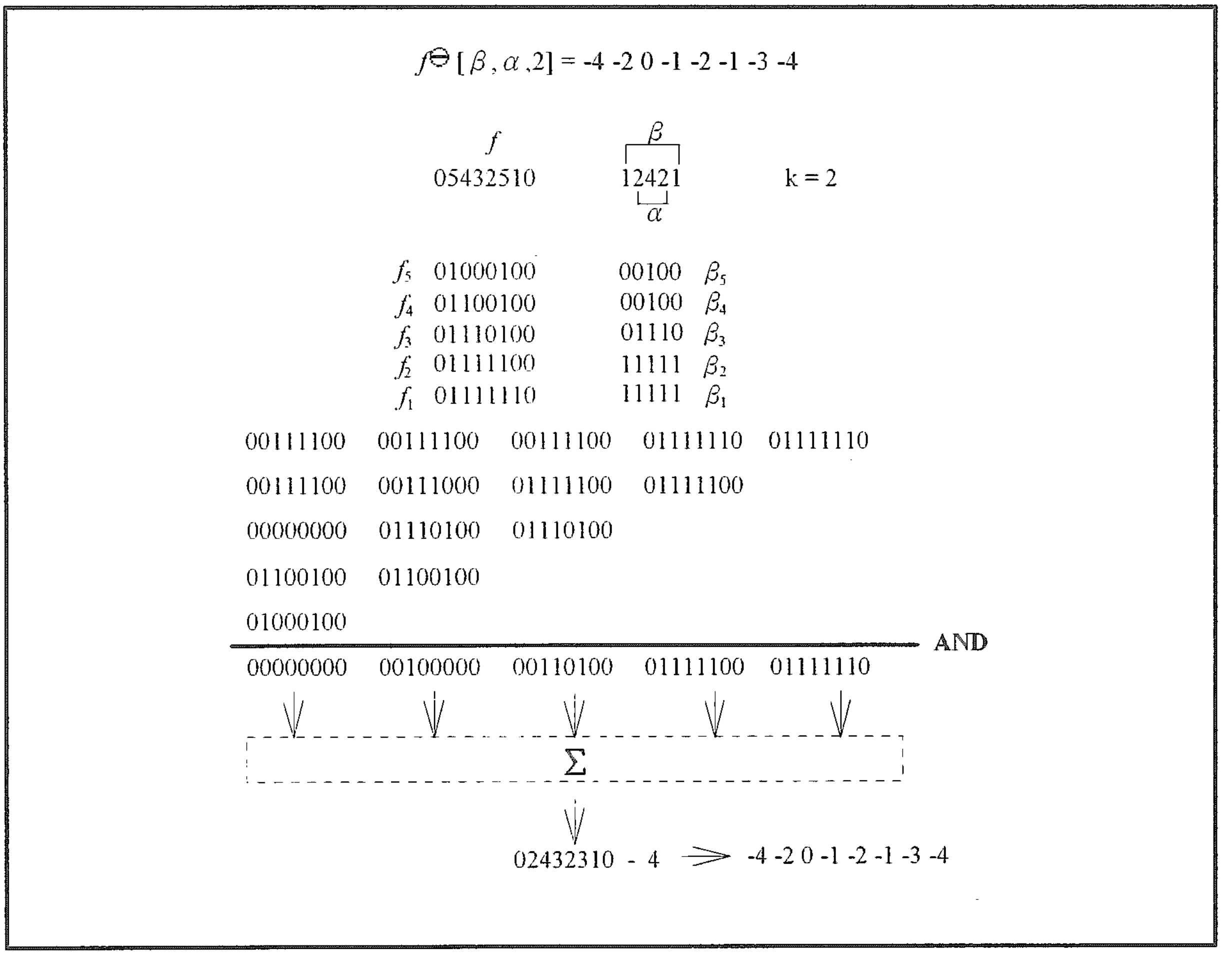

Fig. 4. Threshold decomposition of soft morphological erosion of function-by-function.