Vertex fault tolerance for edge-bipancyclicity of hypercube

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***Abstract*—A bipartite graph $G = (V, E)$ edge-bipancyclic if every edge lies on the cycles of every even length from 4 to $|V|$. Let $Q_n = (V_b \cup V_w, E)$ be an n-dimensional hypercube where V_b and V_w are the sets of black and white vertices, respectively. Let F_b (resp. F_w) be the set of black (resp. white) faulty vertices. In this paper, we will show that $Q_n - F_b - F_w$ is edge-bipancyclic if $|F_b|, |F_w| \leq \lfloor \frac{n-1}{4} \rfloor$.**

***Index Terms*—hypercube, edge-bipancyclic, bipartite graph, vertices fault-tolerance**

I. Introduction

The hypercube network is one of the most popular interconnection networks. It has many attractive properties, such as regularity, symmetry, small degree and diameter, maximum fault tolerance, easy routing algorithms.

An interconnection network is usually represented by a graph where vertices represent processors and edges represent links between processors. Let $G = (V_b \cup V_w, E)$ be a *bipartite graph* where V_b and V_w are two disjoint vertex sets such that each edge of E consists of one vertex from each set. Let $d(u, v)$ be the distance of the vertices u and v. A bipartite graph $G = (V_b \cup V_w, E)$ is *Hamiltonian laceable* if there exists a *Hamiltonian path* between x, y for any $x \in V_b, y \in V_w$. The graph $G = (V_b \cup V_w, E)$ is *hyper-Hamiltonian laceable* if $\forall v \in V_b$(resp. V_w), there exists a Hamiltonian path of $G - \{v\}$ between each pair of vertices of V_w(resp. V_b). In [13], Tsai et al. proved that $Q_n - F_e$ is Hamiltonian laceable for $F_e \subset E(Q_n), |F_e| \leq n - 2$ and hyper-Hamiltonian laceable for $F_e \subset E(Q_n), |F_e| \leq n - 3$. A bipartite graph $G = (V, E)$ is *edge-bipancyclic* if every edge of E lies on cycles of every even length from 4 to $|V|$. In [8], Li et al. proved that $Q_n - F_e$ is edge-bipancyclic for $F_e \subset E(Q_n), |F_e| \leq n - 2$.

There is little literature about general vertex fault tolerant Hamiltonian properties of hypercube $Q_n = (V_b \cup V_w, E)$. Some literatures concern embedding fault-free cycles or paths for hypercube with faulty vertices. The upper bound of longest fault-free cycle of $Q_n - F_v$ is $2^n - 2f$ where F_v is the faulty set of vertices of Q_n and $f = \max\{|F_v \cap V_w|, |F_v \cap V_b|\}$. In [2], [4], [10], [12], the authors showed that a fault-free cycle of length $2^n - 2f_v$ can be constructed with f_v faulty vertices. In [5], [14], the authors showed that every edge in $Q_n - F_v - F_e$ lies on cycles of every even length from 4 to $2^n - 2|F_v|$ if $|F_v| + |F_e| \leq n-2$. When all faulty vertices are in the same partite set, this result is the vertices fault tolerance for edge-bipancyclicity of Q_n. However, there exist longer cycles when both partite vertex sets contain some faulty vertices.

In [1], Caha et al. proposed the multiple spanning paths problem for hypercube. Let s_i, t_i, for $1 \leq i \leq k$, be vertices of Q_n. The $\{s_i, t_i\}_{i=1}^k$ is a *connectable family* if there exists k spanning paths $P(s_i, t_i)$ of Q_n for $1 \leq i \leq k$. The $\{s_i, t_i\}_{i=1}^k$ is *balanced* if it has the same number of vertices in each partite set. Caha showed that every balanced family $\{s_i, t_i\}_{i=1}^n$ is connectable in Q_{2n} if $d(s_i, t_i)$ is odd for $1 \leq i \leq n$. Caha also showed that every balanced family $\{s_i, t_i\}_{i=1}^n$ is connectable in Q_{6n}. In [7], Hung et al. investigated the fault tolerance for connectable family of bipartite graph.

Let the family $\{s_i,t_i\}^{F_b,K_b}_{F_w,K_w}$ of Q_n be the vertex set $K_b \cup K_w = \{s_i,t_i| \text{ for } 1 \le i \le (|K_b|+|K_w|)/2\}$ of $Q_n - F_b - F_w$. The family $\{s_i,t_i\}^{F_b,K_b}_{F_w,K_w}$ is balanced if $2|F_b| + |K_b| = 2|F_w| + |K_w|$. The family $\{s_i,t_i\}^{F_b,K_b}_{F_w,K_w}$ is connectable if there exit $(|K_b|+|K_w|)/2$ spanning disjoint paths $P(s_i,t_i)$ for $1 \le i \le (|K_b|+|K_w|)/2$ of $Q_n - F_b - F_w$. The authors showed that every family $\{s_i,t_i\}^{F_b,K_b}_{F_w,K_w}$ of Q_n is connectable if $|K_b|+|K_w|+|F_b|+|F_w| \le n$ and $4|F_b|+2|K_b| = 4|F_w|+2|K_w| \le n+1$.

In this paper, we incorporate the adjacently faulty vertices into the vertex fault tolerance of multiple spanning paths of hypercube. Let $\{s_i,t_i\}^{F_b,K_b}_{F_w,K_w}$ be a family of $G = (V_b \cup V_w, E) - F_a$ where $K_b \cup K_w = \{s_i,t_i|1 \le i \le (|K_b|+|K_w|)/2\}$ is the set of fault-free vertices, F_a is the set of $|F_a|$ pairs of adjacently faulty vertices, $F_b \subset V_b$ and $F_w \subset V_w$ are sets of faulty vertices. In this paper, we will show that every family $\{s_i,t_i\}^{F_b,K_b}_{F_w,K_w}$ of hypercube $Q_n - F_a$ is connectable if $|F_b|+|F_w|+|K_b|+|K_w|+|F_a| \le n$, $4|F_b|+2|K_b|+|F_a| = 4|F_w|+2|K_w|+|F_a| \le n+1$, for $n \ge 3$. Applying this result, we can obtain that $Q_n - F_b - F_w$ is edge-bipancyclic if $|F_b|, |F_w| \le \lfloor \frac{n-1}{4} \rfloor$.

The rest of this paper is organized as follows. In Section 2, we introduce some important definitions and lemmas. Section 3 shows the vertex fault tolerance for multiple spanning paths. The vertex fault-tolerance for edge-bipancyclicity is introduced in Section 4. We finally give some conclusion in Section 5.

II. Preliminaries

An n-dimensional hypercube $Q_n = (V_b \cup V_w, E)$ is a bipartite graph whose vertices are labeled by distinct n-bit binary strings. Two vertices are linked by an edge if and only if their labels differ exactly in one bit. The hypercube Q_n can be constructed recursively as $Q_n = Q_{n-1} \times K_2$. We can partition Q_n as two subgraphs Q^0_{n-1} and Q^1_{n-1} by choosing any one bit of binary string.

We call the V_b black vertex set and V_w white vertex set. Let V^j_b and V^j_w be the black and white vertex set of Q^j_{n-1} for $j = 0,1$. And let $V^j = V^j_b \cup V^j_w$ for $j = 0,1$. Thus, $V_b = V^0_b \cup V^1_b, V_w = V^0_w \cup V^1_w, V = V_b \cup V_w = V^0 \cup V^1$.

Let F_b be the set of black faulty vertices and F_w be the set of white faulty vertices of Q_n. Similarly, we also use F^j_b and F^j_w to denote the black and white faulty vertex set of Q^j_{n-1}, respectively, for $j = 0,1$. Thus, $F_b = F^0_b \cup F^1_b, F_w = F^0_w \cup F^1_w, F^0 = F^0_b \cup F^0_w, F^1 = F^1_b \cup F^1_w$.

Let F_a be the set of adjacently faulty vertices of Q_n. Similarly, we also use F^j_a to denote the adjacently faulty vertex set of Q^j_{n-1}, respectively, for $j = 0,1$. Thus, $F_a = F^0_a \cup F^1_a$. We further define $F = F_b \cup F_w \cup F_a$.

Let K_b and K_w be the black and white fault-free vertex set. Let $K = K_b \cup K_w = \{s_i,t_i|1 \le i \le \frac{|K|}{2}\}$. And let $K^j_b = K_b \cap V^j, K^j_w = K_w \cap V^j$, for $j = 0,1$.

Let $\phi(v)$ be a vertex of V^i for every $v \in V^j$ such that $(v,\phi(v)) \in E$ and $\{i,j\} = \{0,1\}$. Let $X = \{x_1,x_2,\cdots,x_k\}$ be a vertex subset of Q^i_{n-1} for $i = 0,1$. We define the free neighbor set of X is $N(X) = \{u_j|(x_j,u_j) \in E(Q^i_{n-1}) \text{ and } \phi(u_j) \notin (F \cup K) \text{ for } 1 \le j \le k, i = 0,1\}$. Let $\phi(X) = \{\phi(v)|v \in X\}$ be a vertex subset of V^j for $X \subset V^i$ for $\{i,j\} = \{0,1\}$.

We need some previous results for our proofs. The following lemma is proposed in [6].

Lemma 1: The graph Q_n is f-adjacency $(n-2-f)$ edges Hamiltonian for $0 \le f \le (n-2)$, f-adjacency $(n-2-f)$ edges Hamiltonian laceable for $0 \le f \le (n-3)$, and f-adjacency $(n-3-f)$ edges hyper-Hamiltonian laceable for $0 \le f \le (n-3)$.

A bipartite graph $G = (V_b \cup V_w, E)$ has *property 2H* if for any $s_1,s_2 \in V_b$ and $t_1,t_2 \in V_w$ there exist two spanning disjoint paths $P(s_1,t_1)$ and $P(s_2,t_2)$ of G. Su et al. proved the following lemma in [11].

Lemma 2: The graph $Q_n - F_a - F_e$ has property 2H where F_a is the set of $|F_a|$ pairs adjacently faulty vertices and F_e is the set of faulty edges and $0 \le |F_a| + |F_e| \le n-3$.

III. Vertex fault tolerance for multiple spanning paths in hypercube

In this section, we will prove the vertex fault tolerance for multiple spanning disjoint paths of hypercube. The following lemma is the proof for some property for Q_4.

Lemma 3: Let $s_1, t_1 \in V_w$ and $s_2, t_2 \in V_b$ be two pairs of fault-free vertices. there exist two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of Q_4.

Proof. By symmetry of hypercube, we can arrange s_1 in Q_3^0 and t_1 in Q_3^1. We will prove this lemma in the following cases.

Case 1. s_2 and t_2 in the same subcube.

Without loss of generality, we can assume that s_2, t_2 are in Q_3^1. We can construct a Hamiltonian path $\langle s_2 \overset{P(s_2,t_2)}{\longrightarrow} t_2, x \overset{P(x,t_1)}{\longrightarrow} t_1\rangle$ of Q_3^1. We can also construct a Hamiltonian path $P(s_1, \phi(x))$ of Q_3^0. Thus, $P(s_2, t_2)$ and $\langle s_1 \overset{P(s_1,\phi(x))}{\longrightarrow} \phi(x), x \overset{P(x,t_1)}{\longrightarrow} t_1\rangle$ are two spanning disjoint paths of Q_4.

Case 2. s_2 and t_2 in different subcubes.

Without loss of generality, we can assume that $s_2 \in Q_3^1$ and $t_2 \in Q_3^0$. We can construct a Hamiltonian path $\langle s_1 \overset{P(s_1,x_1)}{\longrightarrow} x_1, x_2 \overset{P(x_2,t_2)}{\longrightarrow} t_2\rangle$ of Q_3^0 for $x_1 \in V_w^0$ and $\{\phi(x_1), \phi(x_2)\} \cap \{s_2, t_1\} = \emptyset$. Applying Lemma 2, we can further construct two spanning disjoint paths $P(\phi(x_1), t_1)$ and $P(s_2, \phi(x_2))$ of Q_3^1. Thus, $\langle s_1 \overset{P(s_1,x_1)}{\longrightarrow} x_1, \phi(x_1) \overset{P(\phi(x_1),t_1)}{\longrightarrow} t_1\rangle$ and $\langle s_2 \overset{P(s_2,\phi(x_2))}{\longrightarrow} \phi(x_2), x_2 \overset{P(x_2,t_2)}{\longrightarrow} t_2\rangle$ are two spanning disjoint paths of Q_4. □

Theorem 1: Every family $\{s_i, t_i\}_{F_w,K_w}^{F_b,K_b}$ of hypercube Q_n is connectable if $|F_b|+|F_w|+|K_b|+|K_w|+|F_a| \leq n$, $4|F_b|+2|K_b|+|F_a| = 4|F_w|+2|K_w|+|F_a| \leq n+1, |F_a| \leq n-3$ for $n \geq 3$.

Proof: We will prove this theorem by induction on n. When $|F_b|+|F_w|+|K_b|+|K_w|+|F_a| < n$, $4|F_b|+2|K_b|+|F_a| = 4|F_w|+2|K_w|+|F_a| < n+1, |F_a| < n-3$, the proof of Q_n is the same as the proof of Q_{n-1}. Thus, we only need to prove that at lest one of the conditions $|F_b|+|F_w|+|K_b|+|K_w|+|F_a| = n$, $4|F_b|+2|K_b|+|F_a| = 4|F_w|+2|K_w|+|F_a| = n+1, |F_a| = n-3$ holds.

Applying Lemma 1, we can obtain that $Q_n - F_a$ is Hamiltonian laceable and hyper-Hamiltonian laceable for $|F_a| = n-3$. Thus, this theorem is true for $|F_a| = n-3$. It is true for $n = 3$.

We consider the case for $n = 4$. Applying Lemma 2, we can also obtain that Q_4 has the property 2H. Applying Lemma 3, we can construct two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of Q_4 for $K_b = \{s_1, t_1\}, K_w = \{s_2, t_2\}$. Thus, this theorem is true for $n = 4$.

We will prove the induction step for $|F_a| \leq n-4$ and $n \geq 5$ with the following cases. By the symmetry of hypercube, we can assume that every pair of adjacently vertices is either in Q_{n-1}^0 or Q_{n-1}^1. We draw Q_{12} in figures of some cases for illustration.

Case 1: $n - |F_a|$ is even.

When $|F_b| + |F_w| \geq 1$, the proof of this case is the same as the case of Q_{n-1}. Thus, we only need to prove this case for $|F_b| = |F_w| = 0$. We can infer that $|K_b| + |K_w| + |F_a| = n$. Without loss of generality, we can assume that $|F_a^0| \geq |F_a^1|$.

Case 1.1: $|F_a| = 0$.

Without loss of generality, we can assume that $|K^0| \geq |K^1| \geq 1$ and $|K_b^0| \geq |K_w^0| \geq 1$.

Case 1.1.1: $|K^1| = 1$.

Without loss of generality, we can assume that $s_2 \in V^1$ and $d(s_1, t_2)$ is odd. Let (s_i, s_i') and (t_i, t_i') be edges of Q_{n-1}^0 for $3 \leq i \leq \frac{|K|}{2}$ such that $t_2 \notin \{\phi(s_i'), \phi(t_i')|$ for $3 \leq i \leq \frac{|K|}{2}\}$. Applying Lemma 1, we can construct a Hamiltonian path $\langle s_1 \overset{P(s_1,t_1)}{\longrightarrow} t_1, x \overset{P(x,t_2)}{\longrightarrow} t_2\rangle$ of $Q_{n-1}^0 - \{s_i, s_i', t_i, t_i' | 3 \leq i \leq \frac{|K|}{2}\}$ for $\phi(x) \neq s_2$. By induction hypothesis, there exist $\frac{|K|}{2} - 1$ spanning disjoint paths $P(s_2, \phi(x))$, $P(\phi(s_i'), \phi(t_i'))$ of Q_{n-1}^1 for $3 \leq i \leq \frac{|K|}{2}$. Thus, $P(s_1, t_1)$, $\langle s_2 \overset{P(s_2,\phi(x))}{\longrightarrow} \phi(x), x \overset{P(x,t_2)}{\longrightarrow} t_2\rangle$, $\langle s_i, s_i', \phi(s_i') \overset{P(\phi(s_i'),\phi(t_i'))}{\longrightarrow} \phi(t_i'), t_i', t_i\rangle$ are $\frac{|K|}{2}$ spanning disjoint paths of Q_n for $3 \leq i \leq \frac{|K|}{2}$, as illustrated in Fig. 1.

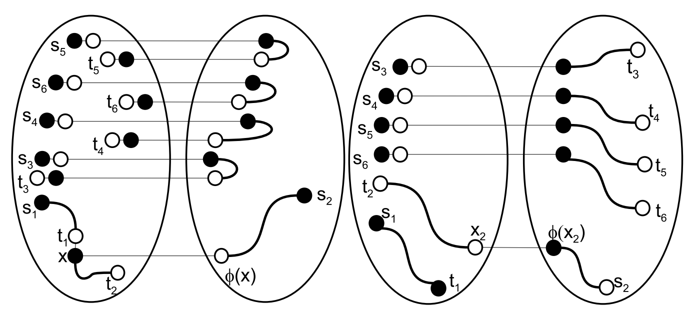

Fig. 1. Illustration of **Case 1.1.1 and Case 1.1.2**

Case 1.1.2: $|K^1| \geq 2$ and $s_i, t_i \in K^0$ for some $1 \leq i \leq \frac{|K|}{2}$.

Without loss of generality, we can assume that $s_1, t_1 \in Q^0_{n-1}$. Let t_2 be vertex of Q^0_{n-1} with $\{s_1, t_1, t_2\} \not\subset K_b$ and $\{s_1, t_1, t_2\} \not\subset K_w$. Without loss of generality, we can assume that $s_1, t_1 \in K_b, t_2 \in K_w$. Let $x_2 \in K^0_w$ for $\phi(x_2) \notin K^1$. Let $K' = K^0 - \{s_1, t_1, t_2\}$, $N(K')$ be the free neighbor set of K'. Applying Lemma 3, we can construct two spanning disjoint paths $P(s_1, t_1)$ and $P(x_2, t_2)$ of $Q^0_{n-1} - K' - N(K')$. By induction hypothesis, there exist $\frac{|K|}{2} - 1$ spanning disjoint paths between $(\phi(N(K')) \cup K^1 \cup \{\phi(x_2)\})$ of Q^1_{n-1}. Therefore, we can construct $\frac{|K|}{2}$ spanning disjoint paths between $K_b \cup K_w$ of Q_n as illustrated in Fig. 1.

Case 1.1.3: s_i and t_i in different subcubes for every $1 \le i \le \frac{|K|}{2}$.

Without loss of generality, we can that $s_i \in Q^0_{n-1}$, and $t_i \in Q^1_{n-1}$ for $1 \le i \le \frac{|K|}{2}$. Suppose that $d(s_i, t_i)$ is odd for some $1 \le i \le \frac{|K|}{2}$. Without loss of generality, we can assume that $d(s_1, t_1)$ is odd. Let (t'_1, t_1) be an edge of Q^1_{n-1} for $t'_1, \phi(t'_1) \notin K$. Let (s_i, s'_i) be edges of Q^0_{n-1} for $s'_i, \phi(s'_i) \notin (K \cup \{t'_1, \phi(t'_1)\})$ for $2 \le i \le \frac{|K|}{2}$. Applying Lemma 1, we can construct a Hamiltonian path $P(s_1, \phi(t'_1))$ of $Q^0_{n-1} - \{s_i, s'_i | 2 \le i \le \frac{|K|}{2}\}$. By the induction hypothesis, there exist $\frac{|K|}{2} - 1$ spanning disjoint paths $P(\phi(s'_i), t_i)$ of $Q^1_{n-1} - \{t_1, t'_1\}$. Therefore, we can construct $\frac{|K|}{2}$ spanning disjoint paths $\langle s_1 \xrightarrow{P(s_1, \phi(t'_1))} \phi(t'_1), t'_1, t_1 \rangle$, $\langle s_i, s'_i, \phi(s'_i) \xrightarrow{P(\phi(s'_i), t'_i)} t'_i \rangle$ of Q_n for $2 \le i \le \frac{|K|}{2}$, as illustrated in Fig. 2.

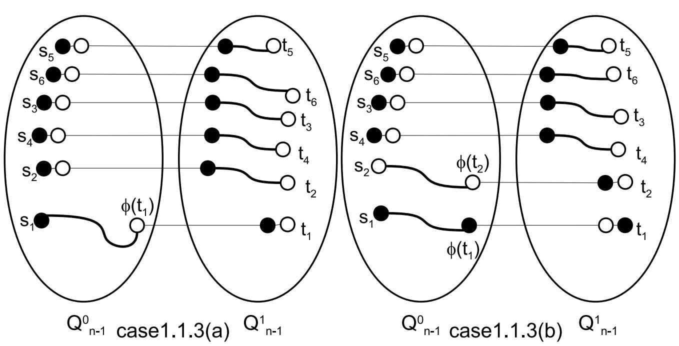

Fig. 2. Illustration of **Case 1.1.3**

Suppose that $d(s_i, t_i)$ is even for every $1 \le i \le \frac{|K|}{2}$. Without loss of generality, we can assume that $s_1 \in K_b$ and $s_2 \in K_w$. Let $(t'_1, t_1), (t'_2, t_2)$ be edges of Q^1_{n-1} for $t'_2, t'_2, \phi(t'_1), \phi(t'_2) \notin K$. Let (s_i, s'_i) be edges of Q^0_{n-1} for $s'_i, \phi(s'_i) \notin (K \cup \{t'_1, t'_2, \phi(t'_1), \phi(t'_2)\})$, for $3 \le i \le \frac{|K|}{2}$. By induction hypothesis, there exist two spanning disjoint paths $P(s_1, \phi(t'_1))$ and $P(s_2, \phi(t'_2))$ of $Q^0_{n-1} - \{s_i, s'_i | 3 \le i \le \frac{|K|}{2}\}$. By induction hypothesis, there also exist $\frac{|K|}{2} - 2$ spanning disjoint paths $P(\phi(s'_i), t_i)$ of $Q^1_{n-1} - \{t_1, t'_1, t_2, t'_2\}$. Therefore, we can construct $\frac{|K|}{2}$ spanning disjoint paths $\langle s_i \xrightarrow{P(s_i, \phi(t'_i))} \phi(t'_i), t'_i, t_i \rangle$, $\langle s_j, s'_j, \phi(s'_j) \xrightarrow{P(\phi(s'_j), t'_j)} t'_j \rangle$ of Q_n for $1 \le i \le 2, 3 \le j \le \frac{|K|}{2}$, as illustrated in Fig. 2(b).

Case 1.2: $|F_a| \ge 1$ and $|F^1_a| = 0$.

Case 1.2.1: $|K^1| = 0$.

We can infer that $|K_b| \ge 2$ and $|K_w| \ge 2$ since $|F_b| = |F_w| = 0$ and $|F_a| \le n - 4$. Without loss of generality, we can assume that $s_1, t_1 \in V_b$ and $s_2, t_2 \in V_w$. Let $(s_i, s'_i), (t'_i, t_i)$ be edges of Q^0_{n-1} for $s'_i, t'_i \notin (K \cup F)$ for $3 \le i \le \frac{|K|}{2}$. Applying Lemma 2, we can construct two spanning disjoint paths $\langle s_1 \xrightarrow{P(s_1, s'_1)} s'_1, t'_2 \xrightarrow{P(t'_2, t_2)} t_2 \rangle$ and $\langle s_2 \xrightarrow{P(s_2, s'_2)} s'_2, t'_1 \xrightarrow{P(t'_1, t_1)} t_1 \rangle$ of $Q^0_{n-1} - F^0_a - \{s_i, s'_i, t_i, t'_i | 3 \le i \le \frac{|K|}{2}\}$. By induction hypothesis, we can construct $\frac{|K|}{2}$ spanning disjoint paths $P(\phi(s'_i), \phi(t'_i))$ of Q^1_{n-1} for $1 \le i \le \frac{|K|}{2}$. Therefore, $\langle s_i \xrightarrow{P(s_i, s'_i)} s'_i, \phi(s'_i) \xrightarrow{P(\phi(s'_i), \phi(t'_i))} \phi(t'_i), t'_i, \xrightarrow{P(t'_i, t_i)} t_i \rangle$, $\langle s_j, s'_j, \phi(s'_j) \xrightarrow{P(\phi(s'_j), \phi(t'_j))} \phi(t'_j), t'_j, t_j \rangle$ are $\frac{|K|}{2}$ spanning disjoint paths of $Q_n - F_a$ for $1 \le i \le 2, 3 \le j \le \frac{|K|}{2}$, as illustrated in Fig. 3.

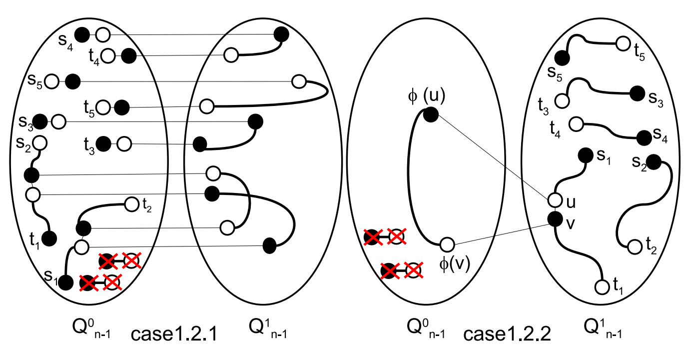

Fig. 3. Illustration of **Case 1.2.1 and Case 1.2.2**

Case 1.2.2: $|K^0| = 0$.

By induction hypothesis,we can construct $\frac{|K|}{2}$ spanning disjoint paths between $K_b \cup K_w$ of Q^1_{n-1}. Without loss of generality, we can denote these paths as $\langle s_1, \xrightarrow{P(s_1, u)} u, v \xrightarrow{P(v, t_1)} t_1 \rangle, P(s_i, t_i)$ for $\phi(u), \phi(v) \notin$

$F_a^0, 2 \leq i \leq \frac{|K|}{2}$. Applying Lemma 1, we can construct a Hamiltonian path $P(\phi(u), \phi(v))$ of $Q_{n-1}^0 - F_a^0$. Therefore, $\langle s_1, \xrightarrow{P(s_1,u)} u, \phi(u) \xrightarrow{P(\phi(u),\phi(v))} \phi(v), v \xrightarrow{P(v,t_1)} t_1 \rangle, P(s_i, t_i)$ are the $\frac{|K|}{2}$ spanning disjoint paths of $Q_n - F_a$ for $2 \leq i \leq \frac{|K|}{2}$, as illustrated in Fig. 3.

Case 1.2.3: $|K^0| = 1$.

Without loss of generality, we can assume that $s_1 \in V_b^0$. Let $x \in V_w^0$ for $x, \phi(x) \notin (K \cup F)$. Applying Lemma 1, we can construct a Hamiltonian path $P(s_1, x)$ of $Q_{n-1}^0 - F_a^0$. By induction hypothesis, we can construct $\frac{|K|}{2}$ spanning disjoint paths $P(\phi(x), t_1), P(s_i, t_i)$ of Q_{n-1}^1 for $2 \leq i \leq \frac{|K|}{2}$. Therefore, $\langle s_1, \xrightarrow{P(s_1,x)} x, \phi(x) \xrightarrow{P(\phi(x),t_1)} t_1 \rangle, P(s_i, t_i)$ are the $\frac{|K|}{2}$ spanning disjoint paths of $Q_n - F_a$, as illustrated in Fig. 4.

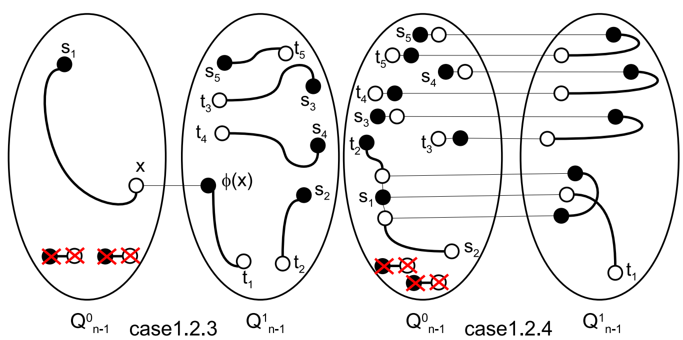

Fig. 4. Illustration of **Case 1.2.3 and Case 1.2.4**

Case 1.2.4: $|K^1| = 1$.

Without loss of generality, we can assume that $t_1 \in V_w^0$. Since $|V_b| \geq 2, |V_w| \geq 2$, we can choose two black vertices and one white vertex of K^0. Without loss of generality, we can assume that $s_1, t_2 \in V_b, s_2 \in V_w$ and $\phi(s_1) \neq t_1$. Let $(s_i, s_i'), (t_i', t_i)$ be edges of Q_{n-1}^0 for $s_i', t_i', \phi(s_i'), \phi(t_i') \notin (K \cup F)$ for $3 \leq i \leq \frac{|K|}{2}$. Applying Lemma 1, we can construct a Hamiltonian path $\langle s_2 \xrightarrow{P(s_2,s_2')} s_2', s_1, t_2' \xrightarrow{P(t_2',t_2)} t_2 \rangle$ of $Q_{n-1}^0 - F_a^0 - \{s_i, s_i', t_i, t_i' | 3 \leq i \leq \frac{|K|}{2}\}$. By induction hypothesis, there exist $\frac{|K|}{2}$ spanning disjoint paths $P(\phi(s_1), t_1), P(\phi(s_i'), \phi(t_i'))$ of Q_{n-1}^1 for $2 \leq i \leq \frac{|K|}{2}$. Therefore, $\langle s_1, \phi(s_1) \xrightarrow{P(\phi(s_1),t_1)} t_1 \rangle$, $\langle s_2 \xrightarrow{P(s_2,s_2')} s_2', \phi(s_2') \xrightarrow{P(\phi(s_2'),\phi(t_2'))} \phi(t_2'), t_2' \xrightarrow{P(t_2',t_2)} t_2 \rangle$, $\langle s_i, s_i', \phi(s_i') \xrightarrow{P(\phi(s_i'),\phi(t_i'))} \phi(t_i'), t_i', t_i \rangle$ are the $\frac{|K|}{2}$ spanning disjoint paths of $Q_n - F_a$ for $3 \leq i \leq \frac{|K|}{2}$, as illustrated in Fig. 4.

Case 1.2.5: $|K^1| \geq 2, |K^0| \geq 2$.

Without loss of generality, we can assume that $s_1, s_2 \in Q_{n-1}^0$. Let $(s_1, s_1'), (s_2, s_2')$ be edges of Q_{n-1}^0 and $s_1', \phi(s_1'), s_2', \phi(s_2') \notin (F \cup K)$. Let $K' = K^0 - \{s_1, s_2\}$ and $N(K')$ be the free neighbor set of K'. Applying Lemma 2, we can construct two spanning disjoint paths $P(s_1, s_1')$ and $P(s_2, s_2')$ of $Q_{n-1}^0 - F_a - K' - N(K')$. By induction hypothesis, we can construct $\frac{|K|}{2}$ spanning disjoint paths between $K^1 \cup \phi(N(K')) \cup \{\phi(s_1'), \phi(s_2')\}$ of Q_{n-1}^1. Therefore, we can construct $\frac{|K|}{2}$ spanning disjoint paths between $K_b \cup K_w$ of $Q_n - F_a$.

Case 1.3: $|F_a^0| \geq 1$ and $|F_a^1| \geq 1$.

Without loss of generality, we can assume that $|K^0| \geq |K^1|$.

Case 1.3.1: $|K^1| = 0$.

there exist $\frac{|K|}{2}$ spanning disjoint paths $\langle s_1, \xrightarrow{P(s_1,u)} u, v \xrightarrow{P(v,t_1)} t_1 \rangle, P(s_i, t_i)$ of $Q_{n-1}^0 - F_a^0$ for $\phi(u), \phi(v) \notin F_a^1, 2 \leq i \leq \frac{|K|}{2}$. Applying Lemma 1, we can construct a Hamiltonian path $P(\phi(u), \phi(v))$ of $Q_{n-1}^1 - F_a^1$. Therefore, $\langle s_1 \xrightarrow{P(s_1,u)} u, \phi(u) \xrightarrow{P(\phi(u),\phi(v))} \phi(v), v \xrightarrow{P(v,t_1)} t_1 \rangle, P(s_i, t_i)$ are $\frac{|K|}{2}$ spanning disjoint paths of $Q_n - F_a$ for $2 \leq i \leq \frac{|K|}{2}$, as illustrated in Fig. 5.

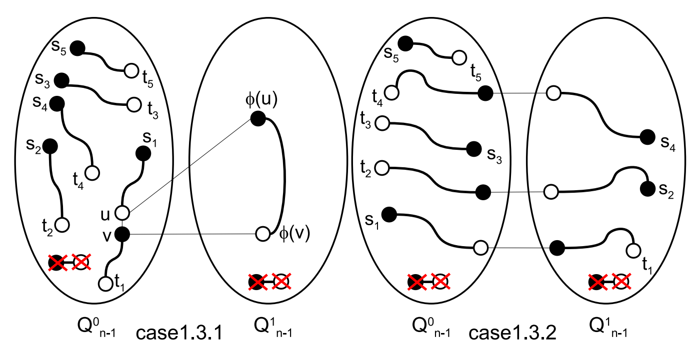

Fig. 5. Illustration of **Case 1.3.1 and Case 1.3.2**

Case 1.3.2: $|K^0| \geq 1$ and $|K^1| \geq 1$

Let $N(K^1)$ be the free neighbor set of K^1. By induction hypothesis, we can construct $\frac{|K|}{2}$ spanning disjoint paths between $K^0 \cup \phi(N(K^1))$ of $Q_{n-1}^0 - F_a^0$ and $|K^1|$ spanning disjoint paths between $N(K^1) \cup K^1$ of $Q_{n-1}^1 - F_a^1$. Therefore, we can construct $\frac{|K|}{2}$ spanning disjoint paths between $K_b \cup K_w$ of $Q_n - F_a$, as illustrated in Fig. 5.

Case 2: $n-|F_a|$ is odd.
When $|F_b|+|F_w|=0$, the proof of this case is the same as Q_{n-1}. Thus, we only need to prove this case for $|F_b|+|F_w|\geq 1$. We can infer that $4|F_b|+2|K_b|+|F_a|=4|F_w|+2|K_w|+|F_a|=n+1$. By symmetry of hypercube, we can assume that $|F_w^0|+|K_w^0|\geq 1$ and $|F_w^1|+|K_w^1|\geq 1$ when $|F_a|=0$. Without loss of generality, we can assume that $4|F^0|+2|K^0|+|F_a^0|\geq 4|F^1|+2|K^1|+|F_a^1|$ and $|F_b^0|\geq|F_w^0|$.

Case 2.1: $4|F_b^0|+|F_a^0|=n+1$.
Since $4|F_b^0|+|F_a^0|=n+1, |F_b^1|=|F_a^1|=|K_b|=0$. Let $b\in F_b^0$ and $F_b'=F_b^1-\{b\}$.

Case 2.1.1: $|K^1|=0$.
Let $(t_1',t_1)\in E(Q_{n-1}^0)$ for $t_1'\notin(F\cup K)$. Let $u_i\in V_b$ be the white vertices of Q^0_{n-1} for $1\leq i\leq 2|F_w^1|$. By induction hypothesis, there exist $\frac{|K|}{2}+|F_w^1|$ spanning disjoint paths $\langle s_1 \xrightarrow{P(s_1,b')} b', b\rangle, P(s_i,t_i), P(u_{2j-1},u_{2j})$ of $Q^0_{n-1}-F_b'-F_w^0-\{t_1,t_1'\}$ for $2\leq i\leq\frac{|K|}{2}, 1\leq j\leq |F_w^1|$. By induction hypothesis, we can also construct $|F_w^1|+1$ spanning disjoint paths $P(\phi(b'),\phi(u_1)), P(\phi(u_{2j}),\phi(u_{2j+1}))$, $P(\phi(u_{2|F_w^1|}),\phi(t_1'))$ of $Q^1_{n-1}-F^1_w$ for $1\leq j\leq |F_w^1|-1$. Therefore, $\langle s_1 \xrightarrow{P(s_1,b')} b',\phi(b') \xrightarrow{P(\phi(b'),\phi(u_1))} \phi(u_1), u_1 \xrightarrow{P(u_1,u_2)} u_2,\phi(u_2),\cdots,u_{2|F_w^1|},\phi(u_{2|F_w^1|}) \xrightarrow{P(\phi(u_{2|F_w^1|}),\phi(t_1'))} \phi(t_1'),t_1',t_1\rangle$, $P(s_i,t_i)$ are $\frac{|K|}{2}$ spanning disjoint paths of $Q_n-(F_b\cup F_w\cup F_a)$ for $2\leq i\leq\frac{|K|}{2}$, as illustrated in Fig. 6.

Case 2.1.2: $|K^0|\geq 1, |K^1|\geq 1$.
Without loss of generality, we can assume that $s_1\in V_w^0$. Let $U=\{u_i|u_i\in V_w^0$ and $u_i,\phi(u_i)\notin(K\cup F)$ for $1\leq i\leq(2|F_w^1|+|K^1|-1)\}$. By induction hypothesis, there exist $(\frac{|K|}{2}+|F_w^1|)$ spanning disjoint paths between $(K^0\cup U\cup\{b_1\})$ of $Q^0_{n-1}-F_b'$. Without loss of generality, we can assume one of these $(\frac{|K|}{2}+|F_w^1|)$ spanning disjoint paths is $\langle s_1 \xrightarrow{P(s_1,b')} b',b\rangle$. By induction hypothesis, we also can construct the $|F_w^1|+|K^1|$ spanning disjoint paths between $K^1\cup\phi(U)\cup\{\phi(b_1')\}$ of $Q^1_{n-1}-F^1_w$. Therefore, we can construct $\frac{|K|}{2}$ spanning disjoint paths between K_w of $Q_n-F_b-F_w-F_a$, as illustrated in Fig. 6.

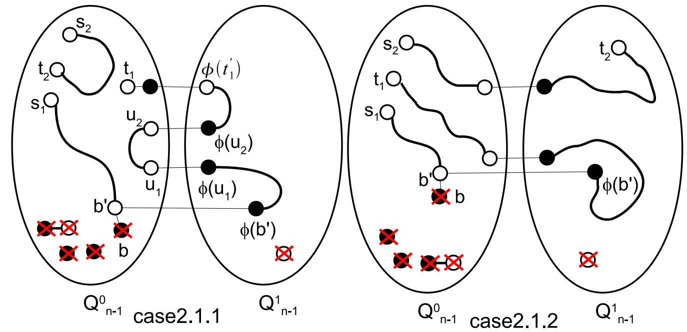

Fig. 6. Illustration of **Case 2.1.1 and Case 2.1.2**

Case 2.1.3: $|K^0|=0$.
Let $(t_1',t_1)\in E(Q^1_{n-1})$ for $t_1',\phi(t_1')\notin(F\cup K)$. Let $u_i\in V_w^0$ for $u_i,\phi(u_i)\notin(F\cup K), 1\leq i\leq 2|F_w^1|+|K|-2$. By induction hypothesis, there exist $(\frac{|K|}{2}+|F_w^1|)$ spanning disjoint paths $\langle b,b' \xrightarrow{P(b',\phi(t_1'))} \phi(t_1')\rangle, P(u_{2i-1},u_{2i})$ of $Q^0_{n-1}-F_b'-F_a^0-F_w^0$ for $1\leq i\leq(\frac{|K|}{2}+|F_w^1|-1)$. By induction hypothesis, we also can construct $(|F_w^1|+|K|-1)$ spanning disjoint paths $P(s_1,\phi(u_1))$, $P(\phi(u_{2i}),\phi(u_{2i+1}))$, $P(\phi(u_{2|F_w^1|+|K|-2}),\phi(b'))$, $P(s_j,t_j)$ of $Q^1_{n-1}-F^1_w-\{t_1,t_1'\}$ for $1\leq i\leq(|F_w^1|+\frac{|K|}{2}-2), 2\leq j\leq\frac{|K|}{2}$. Therefore, $\langle s_1 \xrightarrow{P(s_1,\phi(u_1))} \phi(u_1), u_1 \xrightarrow{P(u_1,u_2)} u_2,\phi(u_2),\cdots, \phi(u_{2|F_w^1|+|K|-2} \xrightarrow{P(\phi(u_{2|F_w^1|+|K|-2}),\phi(b'))} \phi(b'),b' \xrightarrow{P(b',\phi(t_1'))} \phi(t_1'),t_1',t_1\rangle, P(s_j,t_j)$ are $\frac{|K|}{2}$ spanning disjoint paths of $Q_n-F_b-F_w-F_a$ for $2\leq j\leq\frac{|K|}{2}$, as illustrated in Fig. 7.

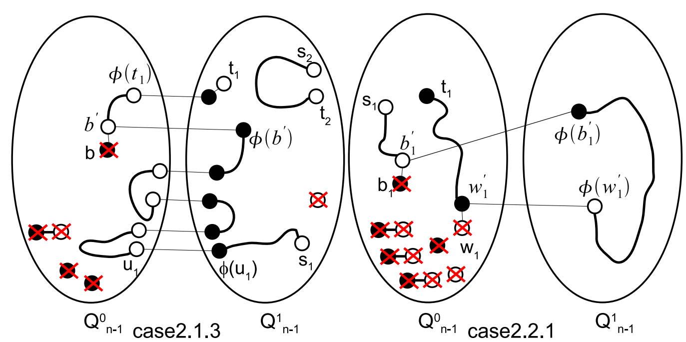

Fig. 7. Illustration of **Case 2.1.3 and Case 2.2.1**

Case 2.2: $4|F_b^0|+|F_a^0|=4|F_w^0|+|F_a^0|=n-1$.
Since $4|F_b^0|+|F_a^0|=4|F_w^0|+|F_a^0|=n-1$, $|F_b^1|=|F_w^1|=|F_a^1|=0$ and $|V_b|=|V_w|=1$. Let $K_w=\{s_1\}$ and $K_b=\{t_1\}$. Let $b_1\in F_b, w_1\in F_w, F_b'=F_b-\{b_1\}$ and $F_w'=F_w-\{w_1\}$. We will construct the Hamiltonian path $P(s_1,t_1)$ of $Q_n-F_b-F_w-F_a$ in the following cases.

Case 2.2.1: $|K^1| = 0$.

By induction hypothesis, we can construct two spanning disjoint paths $\langle s_1 \xrightarrow{P(s_1,b'_1)} b'_1, b_1\rangle$ and $\langle w_1, w'_1 \xrightarrow{P(w'_1,t_1)} t_1\rangle$ of $Q^0_{n-1} - F^0_a - F'_b - F'_w$. Applying Lemma 1, we can obtain a Hamiltonian path $P(\phi(b'_1), \phi(w'_1))$ of Q^1_{n-1}. Thus, $\langle s_1 \xrightarrow{P(s_1,b'_1)} b'_1, \phi(b'_1) \xrightarrow{P(\phi(b'_1),\phi(w'_1))} \phi(w'_1), w'_1 \xrightarrow{P(w'_1,t_1)} t_1\rangle$ is a Hamiltonian path of $Q_n - F_b - F_w - F_a$, as illustrated in Fig. 7.

Case 2.2.2: $|K^1| = 1$.

Without loss of generality, we can assume that $s_1 \in V^0$ and $t_1 \in V^1$. By induction hypothesis, we can construct a Hamiltonian path $\langle s_1 \xrightarrow{P(s_1,w'_1)} w'_1, w_1\rangle$ of $Q^0_{n-1} - F_a - F_b - F'_w$ and a Hamiltonian path $P(\phi(w'_1), t_1)$ of Q^1_{n-1}. Thus, $\langle s_1 \xrightarrow{P(s_1,w'_1)} w'_1, \phi(w'_1) \xrightarrow{P(\phi(w'_1),t_1)} t_1\rangle$ is a Hamiltonian path of $Q_n - F_b - F_w - F_a$, as illustrated in Fig. 8.

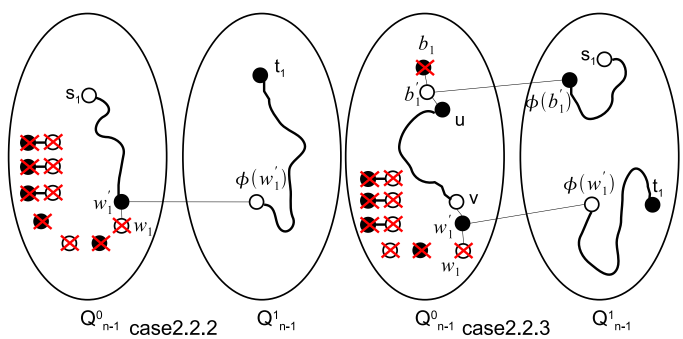

Fig. 8. Illustration of **Case 2.2.2 and Case 2.2.3**

Case 2.2.3: $|K^1| = 2$

Let $(b'_1, b_1), (w'_1, w_1) \in E(Q^0_{n-1})$ for $b'_1, w'_1, \phi(b'_1), \phi(w'_1) \notin (F \cup K)$. Let $(b'_1, u), (v, w'_1) \in E(Q^0_{n-1})$ for $u, v \notin (F_b \cup F_w \cup F_a)$. By induction hypothesis, we can construct a Hamiltonian path $P(u, v)$ of $Q^0_{n-1} - F'_b - F'_w - (F_a \cup \{b_1, b'_1, w_1, w'_1\})$ and two spanning disjoint paths $P(s_1, \phi(b'_1))$ and $P(\phi(w'_1), t_1)$ of Q^1_{n-1}. Thus, $\langle s_1 \xrightarrow{P(s_1,\phi(b'_1))} \phi(b'_1), b'_1, u \xrightarrow{P(u,v)} v, w'_1, \phi(w'_1) \xrightarrow{P(\phi(w'_1),t_1)} t_1\rangle$ is a Hamiltonian path of $Q_n - F_a - F_b - F_w$, as illustrated in Fig. 8.

Case 2.3: $4|F^0_b| + |F^0_a| \leq n - 1$ and $4|F^0_w| + |F^0_a| \leq n - 3$ and $|K^1| = 0$.

Case 2.3.1: $|F^1_a| = 0$ and $|F^1| = 0$.

Since $4|F^0_b| + |F^0_a| \leq n - 1$ and $|K^1| = 0, |K^0_b| \geq 1$. Without loss of generality, we can assume that $t_1 \in V_b$. Let $(t'_1, t_1) \in E(Q^0_{n-1})$ for $t'_1 \notin (K \cup F)$. By induction hypothesis, there exist $\frac{|K|}{2} - 1$ spanning disjoint paths $P(s_i, t_i)$ for $2 \leq i \leq \frac{|K|}{2}$ of $Q^0_{n-1} - F^0 - F^0_a - \{t_1, t'_1\}$. Without loss of generality, we can assume that s_1 is on the path $P(s_2, t_2)$. We can denote $P(s_2, t_2)$ as $\langle s_2 \xrightarrow{P(s_2,u)} u, s_1, v \xrightarrow{P(v,t_2)} t_2\rangle$. By induction hypothesis, we can construct two spanning disjoint paths $P(\phi(s_1), \phi(t'_1))$ and $P(\phi(u), \phi(v))$ of Q^1_{n-1}. Therefore, $\langle s_1, \phi(s_1) \xrightarrow{P(\phi(s_1),\phi(t'_1))} \phi(t'_1), t'_1, t_1\rangle$, $\langle s_2 \xrightarrow{P(s_2,u)} u, \phi(u) \xrightarrow{P(\phi(u),\phi(v))} \phi(v), v \xrightarrow{P(v,t_2)} t_2\rangle$, $P(s_i, t_i)$ are $\frac{|K|}{2}$ spanning disjoint paths of $Q^1_{n-1} - F_b - F_w - F_a$ for $3 \leq i \leq \frac{|K|}{2}$, as illustrated in Fig. 9.

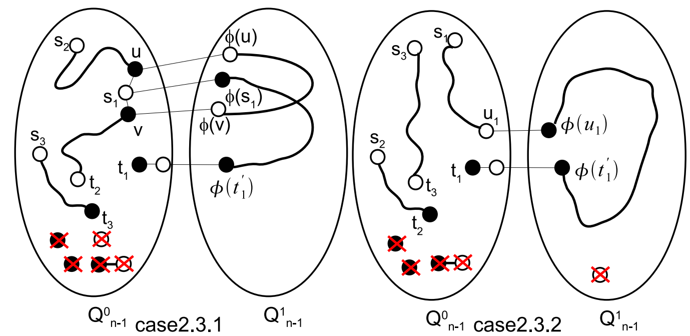

Fig. 9. Illustration of **Case 2.3.1 and Case 2.3.2**

Case 2.3.2: $|F^1_a| = 0$ and $|F^1| \geq 1$ and ($|F^1_b| = 0$ or $|F^1_w| = 0$).

Without loss of generality, we can assume that $|F^1_b| = 0$ and $t_1 \in V^0_b$. Let $(t'_1, t_1) \in E(Q^0_{n-1})$ for $t'_1 \notin (K \cup F)$. Let $U = \{u_i | u_i \in V^0_w$ and $u_i, \phi(u_i) \notin (K \cup F)$ for $1 \leq i \leq (2|F^1_w| - 1)\}$. By induction hypothesis, there exist $(\frac{|K|}{2} + |F^1_w| - 1)$ spanning disjoint paths $P(s_1, u_1), P(u_{2i}, u_{2i+1}), P(s_j, t_j)$ of $Q^0_{n-1} - F^0_b - F^0_w - F^0_a - \{t_1, t'_1\}$ for $1 \leq i \leq |F^1_w| - 1, 2 \leq j \leq \frac{|K|}{2}$. By induction hypothesis, we also can construct the $|F^1_w|$ spanning disjoint paths $P(\phi(u_{2i-1}), \phi(u_{2i})), P(\phi(u_{2|F^1_w|-1}, \phi(t'_1))$of $Q^1_{n-1} - F^1_w$ for $1 \leq i \leq |F^1_w| - 1$. Therefore, $\langle s_1 \xrightarrow{P(s_1,u_1)} u_1, \phi(u_1) \xrightarrow{P(\phi(u_1),\phi(u_2))} \phi(u_2), \cdots, \phi(u_{2|F^1_w|-1} \xrightarrow{P(\phi(u_{2|F^1_w|-1}),\phi(t'_1))} \phi(t'_1), t'_1, t_1\rangle, P(s_j, t_j)$ are $\frac{|K|}{2}$ spanning disjoint paths of $Q_n - F_b - F_w - F_a$ for $2 \leq j \leq \frac{|K|}{2}$, as illustrated in Fig. 9.

Case 2.3.3: $|F_a^1| \geq 1$ or $|F_w^1| = |F_b^1| \geq 1$.
By induction hypothesis, there exist $\frac{|K|}{2}$ spanning disjoint paths $P(s_i, t_i)$ of $Q_{n-1}^0 - F_b^0 - F_w^0 - F_a^0$ for $1 \leq i \leq \frac{|K|}{2}$. Without loss of generality, we can assume that $P(s_1, t_1) = \langle s_1 \xrightarrow{P(s_1,u)} u, v \xrightarrow{P(v,t_1)} t_1\rangle$ for $\phi(u), \phi(v) \notin F_a^1$. Applying Lemma 1, we can construct a Hamiltonian path $P(\phi(u), \phi(v))$ of $Q_{n-1}^1 - F_a^1$. Therefore, $\langle s_1 \xrightarrow{P(s_1,u)} u, \phi(u) \xrightarrow{P(\phi(u),\phi(v))} \phi(v), v \xrightarrow{P(v,t_1)} t_1\rangle, P(s_i, t_i)$ are $\frac{|K|}{2}$ spanning disjoint paths of $Q_n - F_b - F_w - F_a$ for $2 \leq i \leq \frac{|K|}{2}$, as illustrated in Fig. 10.

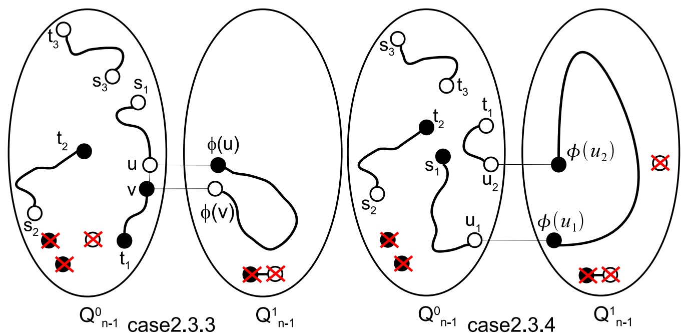

Fig. 10. Illustration of **Case 2.3.3 and Case 2.3.4**

Case 2.3.4: $|F_a^1| + |F_b^1| \geq 1$ and $|F_a^1| + |F_w^1| \geq 1$ and $|F_b^1| \neq |F_w^1|$.
Without loss of generality, we can assume that $|F_w^1| \geq |F_b^1|$. Let $m = |F_w^1| - |F_b^1|$. Let $U = \{u_i | u_i \in V_w^0$ and $u_i, \phi(u_i) \notin (K \cup F)$ for $1 \leq i \leq 2m\}$. By induction hypothesis, there exist $(\frac{|K|}{2} + m)$ spanning disjoint paths $P(s_1, u_1), P(u_{2i}, u_{2i+1}), P(u_{2m}, t_1), P(s_j, t_j)$ of $Q_{n-1}^0 - F_b^0 - F_w^0 - F_a^0$ for $1 \leq i \leq m-1, 2 \leq j \leq \frac{|K|}{2}$. By induction hypothesis, we also can construct the m spanning disjoint paths $P(\phi(u_{2i-1}), \phi(u_{2i}))$of $Q_{n-1}^1 - F_w^1 - F_b^1 - F_a^1$ for $1 \leq i \leq m$. Therefore, $\langle s_1 \xrightarrow{P(s_1,u_1)} u_1, \phi(u_1) \xrightarrow{P(\phi(u_1),\phi(u_2))} \phi(u_2), \cdots, \phi(u_{2m-1} \xrightarrow{P(\phi(u_{2m-1}),\phi(u_{2m}))} \phi(u_{2m}), u_{2m} \xrightarrow{P(u_{2m},t_1)} t_1\rangle, P(s_j, t_j)$ are $\frac{|K|}{2}$ spanning disjoint paths of $Q_n - F_b - F_w - F_a$ for $2 \leq j \leq \frac{|K|}{2}$, as illustrated in Fig. 10.
Case 2.4: $4|F_b^0| + |F_a^0| \leq n-1$ and $4|F_w^0| + |F_a^0| \leq n-3$, $|K^1| \geq 1$.
Case 2.4.1: $|K_b^1| + |F_b^1| + |F_a^1| = 0$ or $|K_w^1| + |F_w^1| + |F_a^1| = 0$.
Without loss of generality, we can assume that $|K_b^1| + |F_b^1| + |F_a^1| = 0$. Let b be a faulty vertex of F_b^0. Let $m = |K_w^1| + 2|F_w^1|$. Let $U = \{u_i | u_i \in V_w^0, u_i \notin (K^0 \cup F^0 \cup F_a^0)$ for $1 \leq i \leq m-1\}$. By induction hypothesis, there exist $\frac{|K|}{2}$ spanning disjoint paths between $K^0 \cup U \cup \{b\}$ of $Q_{n-1}^0 - F_a - F_w - (F_b - \{b\})$ where $P(s_1, b)$ is one of these spanning disjoint paths. We can denote $P(s_1, b)$ as $\langle s_1 \xrightarrow{P(s_1,b')} b', b\rangle$. By induction hypothesis, we can construct $|K_w^1| + |F_w^1|$ spanning disjoint paths between $\phi(U) \cup K_w^1 \cup \{\phi(b')\}$ of $Q_{n-1}^1 - F_w^1$. Thus, we can construct $\frac{|K|}{2}$ spanning disjoint paths of $Q_n - F_b - F_w - F_a$, as illustrated in Fig. 11.

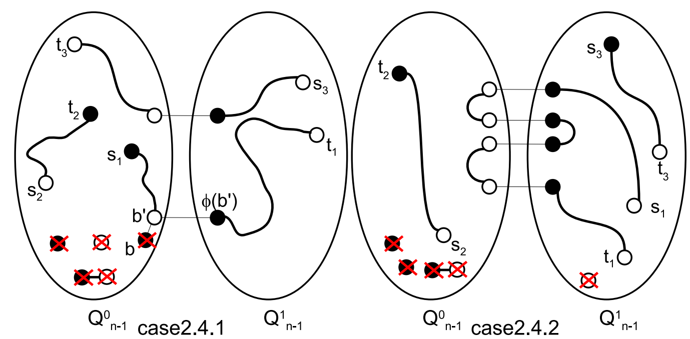

Fig. 11. Illustration of **Case 2.4.1 and Case 2.4.2**

Case 2.4.2: $|K_b^1| + |F_b^1| + |F_a^1| \geq 1$ and $|K_w^1| + |F_w^1| + |F_w^1| \geq 1$.
Without loss of generality, we can assume that $2|F_b^0| + |K_b^0| \geq 2|F_w^0| + |K_w^0|$. Let $m = 2|F_b^0| + |K_b^0| - 2|F_w^0| - |K_w^0|$. Let $X = \{[s_i, t_i] | s_i$ and t_i in different subcubes $\}$ and $|X|$ be the number of pairs of X. Suppose that $m \geq |X|$. Let $U_w = \{u_i | u_i \in V_w^0$, for $1 \leq i \leq m\}$ and $U_b = \emptyset$. Suppose that $m < |x|$. Let $U_w = \{u_i | u_i \in V_w^0$, for $1 \leq i \leq \frac{|X|+m}{2}\}$ and $U_b = \{u_i | u_i \in V_b^0$, for $1 + \frac{|X|+m}{2} \leq i \leq |X|$. By induction hypothesis, there exist $\frac{|K^0|+|U_w|+|U_b|}{2}$ spanning disjoint paths between $K^0 \cup U_w \cup U_b$ of $Q_{n-1}^0 - F_b^0 - F_w^0 - F_a^0$ and $\frac{|K^1|+|U_w|+|U_b|}{2}$ spanning disjoint paths of between $K^1 \cup \phi(U_b) \cup \phi(U_w)$ of $Q_{n-1}^1 - F_b^1 - F_w^1 - F_a^1$. Therefore, we can construct $\frac{|K|}{2}$ spanning disjoint paths between of $Q_n - F_b - F_w - F_a$, as illustrated in Fig. 11. □

IV. VERTICES FAULT-TOLERANCE FOR EDGE-BIPANCYCLICITY OF HYPERCUBE

In this section, we prove the vertices fault-tolerance for edge bipancyclicity of hypercube. The following lemma is proved in [4].

Lemma 4: Every edge in $Q_n - F_v - F_e$ lies on a cycle of every even length from 4 to $2^n - 2|F_v|$ even if $|F_v| + |F_e| \leq n - 2$, for $n \geq 3$.

Theorem 2: Let F_b and F_w be the sets of faulty black vertices and faulty while vertices, respectively, of hypercube Q_n. The graph $Q_n - F_b - F_w$ is edge-bipancyclic if $|F_b|, |F_w| \leq \lfloor \frac{n-1}{4} \rfloor$ for $n \geq 3$.

Proof: Let $e = (s, t)$ be an arbitrary edge of $Q_n - F_b - F_w$ for $s \in V_b$. Applying Lemma 4, we can obtain that there exist cycle containing the edge e with even length from 4 to $2^n - 2(|F_b| + |F_w|)$ of $Q_n - F_b - F_w$. Let $F_b = \{b_1, b_2, \cdots, b_{f_1}\}$ and $F_w = \{w_1, w_2, \cdots, w_{f_2}\}$. Without loss of generality, we can assume that $f_1 \geq f_2$. Let $F_a = \{b_i, x_i|$ for $(b_i, x_i) \in E(Q_n)$ and $x_i \notin (F_b \cup F_w \cup \{s, t\})$ for $f_2 + 1 \leq i \leq f_1\}$ and $|F_a|$ be the number of pair of adjacently vertices of F_a. Let $F_{a_j} = \{b_i, x_i, w_i, y_i|$ for $(b_i, x_i), (w_i, y_i) \in E(Q_n)$ and $x_i, y_i \notin (F_b \cup F_w \cup \{s, t\})$ for $j \leq i \leq f_2\}$ for $1 \leq j \leq f_2$ and $|F_{a_j}|$ be the number of pair of adjacently vertices of F_{a_j}. Let $F'_b = \{b_1, b_2, \cdots, b_{f_2}\}$. We can check that $|F_a| + |F_w| + |F'_b| + |F_{a_j}| + 2 = f_1 + f_2 + 2 \leq \frac{n+3}{2} < n$ and $4|F'_b| + 2 + |F_a| + |F_{a_j}| = 4|F_w| + 2 + |F_a| + |F_{a_j}| \leq n + 1$ for $1 \leq j \leq f_2$. Applying Theorem 1, we can construct a Hamiltonian path $P(s, t)$ of $Q_n - F'_b - F_w - F_a - F_{a_j}$ for $1 \leq j \leq f_2$. Thus, we can construct the cycles $\langle s \xrightarrow{P(s,t)} t, s \rangle$ containing the edge e with even length from $2^n - 2(|F_b| + |F_w|)$ to $2^n - 2\max\{|F_b|, |F_w|\}$ of $Q_n - F_b - F_w$. Therefore, $Q_n - F_b - F_w$ is edge-bipancyclic. □

V. Conclusion

In this paper, we show that every family $\{s_i, t_i\}_{F_w, K_w}^{F_b, K_b}$ of hypercube $Q_n - F_a$ is connectable if $|F_b| + |F_w| + |K_b| + |K_w| + |F_a| \leq n$, $4|F_b| + 2|K_b| + |F_a| = 4|F_w| + 2|K_w| + |F_a| \leq n + 1$, for $n \geq 3$. Applying this result, we show that $Q_n - F_b - F_w$ is edge-bipancyclic if $|F_b|, |F_w| \leq \lfloor \frac{n-1}{4} \rfloor$.

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