On Distance-Two Domination of Composition of Graphs

Yung-Ling Lai
Department of Computer Science and Information Engineering, National Chiayi University, Taiwan
Email: yllai@mail.ncyu.edu.tw

Shou-Bo Jeng
Department of Computer Science and Information Engineering, National Chiayi University, Taiwan
Email: s0970393@mail.ncyu.edu.tw

***Abstract*—For a graph $G=(V,E)$, let $N_1(v)$ and $N_2(v)$ denote the set of vertices that are at distance one and two from v respectively. A subset $D \subseteq V(G)$ is said to be a $D_{3,2,1}$-dominating set of G if every vertex $v \in V$ satisfies $w_D(v) \geq 3$ where $w_D(v) = 3\,|\{v\} \cap D| + 2\,|N_1(v) \cap D| + |N_2(v) \cap D|$. The minimum cardinality of a $D_{3,2,1}$-dominating set of G, denoted as $\gamma_{3,2,1}(G)$, is called the $D_{3,2,1}$-domination number of G. In this paper we obtained the $D_{3,2,1}$-domination number of the composition of two paths and a path with a cycle.**

***Index Terms*— $D_{3,2,1}$-domination, composition, $D_{3,2,1}$-domination number.**

I. Introduction

We consider only simple and connected graphs. A graph $G=(V,E)$ contains a set V of vertices and a set E of edges. The *distance* $d(x,y)$ of two vertices x and y is the length of the shortest $x-y$ path. The *distance-k-neighborhood* $N_k(v)$ of vertex v, defined as $N_k(v)=\{\, u \in v \mid d(u,v)=k \,\}$, is the set of those vertices that are at distance k from v. Figure 1 shows an example of a graph G with $V=\{a,b,c,d,e,f\}$ where $N_1(a)=\{b,d\}$, $N_2(a)=\{c,e,f\}$. For a graph $G=(V,E)$, a *dominating set* $D \subseteq V$ *of* G is a set of vertices such that for each $u \in V-D$, $N_1(u) \cap D \neq \varnothing$. The *distance-k-dominating set* of a graph G is defined as the subset $D \subseteq V$ such that for each $u \in V-D$, $\bigcup_{1 \leq i \leq k} (N_i(u) \cap D) \neq \varnothing$. The $D_{3,2,1}$*-domination problem* proposed by [12] in 2006 is similar to distance-2-domination problem, which may be used to solved the resource sharing problem that are modeled by graphs. For each vertex v, the *weight of* v is defined as $w_D(v)=3|\{v\} \cap D| + 2|N(v) \cap D| + |N_2(v) \cap D|$ for some $D \subseteq V$. D is called a $D_{3,2,1}$*-dominating set of graph* G if and only if for each $v \in V(G)$, $w_D(v) \geq 3$. The $D_{3,2,1}$ *-domination number* $\gamma_{3,2,1}(G)$ *of a graph* G is then the minimum cardinality among all $D_{3,2,1}$-dominating set of G. D is *an optimal* $D_{3,2,1}$ *-dominating set of* G if $|D|=\gamma_{3,2,1}(G)$. Due to the short history, unlike the related distance-k-domination has many results (see [1,3-5,7-9,13,16] for $k=1$, and [2,6,10-11,14-15] for $k>1$), $D_{3,2,1}$-domination problem has only been solved for a very limited class of graphs. The $D_{3,2,1}$-domination number is known for paths, cycles and a full binary tree B_n [12]. [17] discussed $D_{3,2,1}$-domination problem of a *double-loop network* $DL(n;a,b)$ according to different value of a,b. This paper established the $D_{3,2,1}$-domination number for the composition of two paths and a path with a cycle.

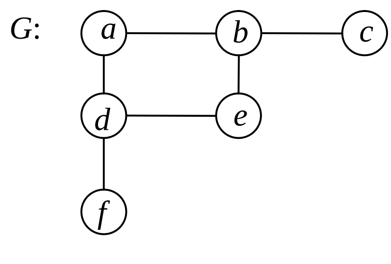

Figure 1 $N_1(a)=\{b,d\}$, $N_2(a)=\{c,e,f\}$

II. RESULTS

The *composition* (also called *lexicographic product*) $G[H]$ of two graphs G and H with vertex set $V(G)=\{v_x|1\le x\le n\}$ and $V(H)$ = $\{u_x|1\le x\le m\}$ respectively, is the graph with vertex set $V(G[H])=V(H)\times V(G)$ and (u_1,v_1) is adjacent to (u_2,v_2), if either v_1 is adjacent to v_2 in G or $v_1=v_2$ and u_1 is adjacent to u_2 in H. Figure 3 shows an example of a $P_3[P_4]$. In this paper, we will use $c_i=\{(u_j,v_i)|1\le j\le m\}$ for $1\le i\le n$ to denote the set of vertices from *i-th column* of $G[H]$.

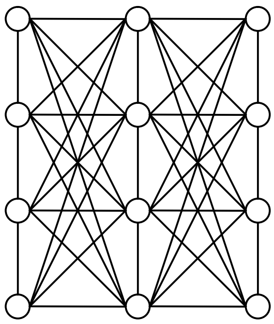

Figure 3: An example of graph $P_3[P_4]$.

Since $D_{3,2,1}$-domination requires every vertex has weight at least 3, Fact 1 is trivial.

Fact 1: *Let* G *be a graph of order* $n\ge 2$. *Then* $\gamma_{3,2,1}(G)\ge 2$.

Lemma 1: Let $G=P_n[P_m]$ for $n>10, m>6$ and D be an optimal $D_{3,2,1}$-dominating set of G. Then in any four consecutive columns of G, there is at least one vertex in D.

Proof: Suppose to the contrary that there is no vertex in D in four consecutive columns $c_i, c_{i+1}, c_{i+2}, c_{i+3}$. In order to have all vertices $v\in c_{i+1}\bigcup c_{i+2}$ satisfy $w_D(v)\ge 3$, there must be at least three vertices of D in c_{i-1} and c_{i+4}. In this case, the best possible is using six vertices to dominate ten columns (from c_{i-3} to c_{i+6}). If n=11 or n=12, then we must have $|D|\ge 7$, but there exist a $D_{3,2,1}$-dominating set $D'=\{(u_1,v_2),(u_1,v_3),(u_1,v_6),(u_1,v_7),(u_1,v_{10}),(u_1,v_{11})\}$ such that $|D'|=6<7\le|D|$, contradict to the fact that D is optimal. If n>12, we discuss in following cases:

Case 1: $|c_{i+7}\cap D|=n(n\ne 0)$, there exist a $D_{3,2,1}$-dominating set $D'=\{(u_1,v_{i\text{-}2}),(u_1,v_{i\text{-}1}),(u_1,v_{i+3}),(u_2,v_{i+3}),(u_1,v_{i+5})\}\cup\{D\text{-}\bigcup_{k=i\text{-}3}^{i+6}(D\cap c_k)\}$. Since $|\bigcup_{k=i\text{-}3}^{i+6}(D\cap c_k)|$=6, we must have $|D'|<|D|$, which contradict to the fact that D is optimal.

Case 2: $|c_{i+8}\cap D|=n(n\ne 0)$ and $|c_{i+7}\cap D|=0$, there exist a $D_{3,2,1}$-dominating set $D'=\{(u_1,v_{i\text{-}2}),(u_1,v_{i\text{-}1}),(u_1,v_{i+3}),(u_2,v_{i+3}),(u_1,v_{i+5})\}\cup\{D\text{-}\bigcup_{k=i\text{-}3}^{i+7}(D\cap c_k)\}$. Since $|\bigcup_{k=i\text{-}3}^{i+7}(D\cap c_k)|$=6, we must have $|D'|<|D|$, which contradict to the fact that D is optimal.

Case 3: $|\{c_{i+7},c_{i+8}\}\cap D|=0$, in order to fulfill the condition $w_D(v)\ge 3$ for all $v\in\{c_{i+7},c_{i+8}\}$, we must have $|c_{i+9}\cap D|=3$, there exist a $D_{3,2,1}$-dominating set $D'=\{(u_1,v_{i\text{-}2}),(u_1,v_{i\text{-}1}),(u_1,v_{i+2}),(u_1,v_{i+3}),(u_1,v_{i+6}),(u_1,v_{i+7}),(u_1,v_{i+10}),(u_1,v_{i+11})\}\cup\{D\text{-}\bigcup_{k=i\text{-}3}^{i+11}(D\cap c_k)\}$. Since $|\bigcup_{k=i\text{-}3}^{i+11}(D\cap c_k)|$=9, we must have $|D'|<|D|$, which contradict to the fact that D is optimal. ■

Lemma 2: *Let* D *be an optimal* $D_{3,2,1}$*-dominating set of* $G=P_n[P_m]$ *for* $m>6$ *and* n *is even. If* $c_1\cap D\ne\varnothing$ *or* $c_n\cap D\ne\varnothing$*, then* $|D|>n/2$.

Proof: Without loss of generality, assume $c_1\cap D\ne\varnothing$. Consider following cases:

Case 1: $|c_1\cap D|=1$. In order to fulfill the condition $w_D(v)\ge 3$ for all $v\in c_1$, we must have either $|c_2\cap D|\ge 1$ or $c_2\cap D=\varnothing$ and $|c_3\cap D|\ge 2$.

Subcase 1.1: $|c_2 \cap D| = 1$. Assume that from c_1 through c_4, there are no more than two vertices in D. Since the vertices in c_4 can not get enough weight from those two vertices, in order to fulfill the condition $w_D(v) \geq 3$ for all $v \in c_4$, we must have either $|c_5 \cap D| \geq 1$ or $|c_6 \cap D| \geq 2$.

If $|c_5 \cap D| = 1$, the sub graph from c_5 to c_n is the same as G in case 1.

If $|c_5 \cap D| \geq 2$, then the case from c_5 to c_n is the same as G in case 2.

If $|c_6 \cap D| = 2$, since the vertices in c_6 are distance 4 from c_2, hence the vertices in c_6 can only get weight from $c_6 \cap D$. Since $m > 6$, there must be some vertices in c_6 without enough weight from those two vertices in c_6. Hence it is impossible for those four vertices to $D_{3,2,1}$-dominate c_1 through c_8.

If $|c_6 \cap D| = y > 2$, assume that from c_1 through c_{2y+4}, there are no more than $y+2$ vertices in D. Since the vertices in c_{2y+3} are at least distance $2y-3$ from c_6, and $y \geq 3$, the vertices in c_{2y+3} can not get any weight from the vertices in c_1 through c_{2y+4}. Then we must have $|c_{2y+5} \cap D| \geq 3$, which again will bring us to case 2.

Subcase 1.2: $|c_2 \cap D| = x > 1$. Assume that from c_1 through c_{2x+2}, there are no more than $x+1$ vertices in D. Since the vertices in c_{2x+1} are at least distance $2x-1$ from c_2, and $x \geq 2$, hence the vertices in c_{2x+1} can not get any weight from the vertices in c_1 through c_{2x+2}, then we must have $|c_{2x+3} \cap D| \geq 3$, which again will bring us to case 2.

Subcase 1.3: $c_2 \cap D = \varnothing$ and $|c_3 \cap D| = x \geq 2$. Assume that from c_1 through c_{2x+2}, there are no more than $x+1$ vertices in D. The vertices in c_{2x+1} are at least distance $2x-2$ from c_3. If $x = 2$, the vertices in c_5 are distance 2 from c_3, and $|c_3 \cap D| = 2$, hence the weight of the vertices in c_5 can only get two from the vertices in c_1 through c_6. Then we must have $|c_7 \cap D| \geq 1$. If $|c_7 \cap D| = 1$, then the case from c_7 to c_n is the same as G in case1. If $|c_7 \cap D| \geq 2$, then the case from c_7 to c_n is the same as G in case 2. If $x \geq 3$, since the vertices in c_{2x+1} are at least distance $2x-2$ from c_3, the vertices in c_{2x+1} can not get any weight from the vertices in c_1 through c_{2x+2}, then we must have $|c_{2x+3} \cap D| \geq 3$, which again will bring us to case 2.

Case 2: $|c_1 \cap D| \geq 2$. Consider following subcases:

Subcase 2.1: $|c_1 \cap D| = 2$. Assume that from c_1 through c_4, there are no more than two vertices in D. Since the vertices in c_3 are distance 2 from c_1, the vertices in c_3 can only get two weight from the vertices in c_1 through c_4, then we must have $|c_5 \cap D| \geq 1$. If $|c_5 \cap D| = 1$, then the case from c_5 to c_n is the same as G in case1. If $|c_5 \cap D| \geq 2$, then the case from c_5 to c_n is the same as G in case2.

Subcase 2.2: $|c_1 \cap D| = x > 2$. Assume that from c_1 through c_{2x}, there are no more than x vertices in D. Since the vertices in c_{2x-1} are at least distance $2x-2$ from c_1, and $x \geq 3$, hence the vertices in c_{2x-1} can not get any weight from the vertices in c_1 through c_{2x}. In order to fulfill the condition $w_D(v) \geq 3$ for all $v \in c_{2x-1}$, we must have $|c_{2x+1} \cap D| \geq 3$, then the case from c_{2x+1} to c_n is the same as G in case2.

By all the cases above, we know that if n is even, for any optimal $D_{3,2,1}$-dominating set D of $P_n[P_m]$, either $(c_1 \cup c_n) \cap D = \varnothing$, or $|D| > n/2$. ■

Algorithm 1: A way to "partition" the graph $P_n\left[P_m\right]$ by the vertices in the $D_{3,2,1}$-dominating set, which is used in the proof of Theorem 1.

Input: Integers *n*, *m*, and a $D_{3,2,1}$-dominating set D with $\left|D\right|<n/2$ of $P_n\left[P_m\right]$.

Task: Find $\max k$, $\min k'$ $\left(1\le k\le k'\le n\right)$, D_1 and D_2 such that $D_1=\bigcup_{i=1}^{k-1}c_i\cap D, \left|D_1\right|=\frac{k-1}{2}$, $D_2=\bigcup_{i=k'+1}^{n}c_i\cap D, \left|D_2\right|=\frac{n-k'}{2}$. Notice that if such k exists, then for odd *n*, $k=k'$; and for even n, $k+1=k'$.

Method:

$k\leftarrow 0$; $x\leftarrow 0$; $k'\leftarrow 0$; $y\leftarrow 0$; $z\leftarrow 0$;
$D_1=\varnothing$; $D_2=\varnothing$;
if (n % 2 = 0) $z\leftarrow 2$;
else $z\leftarrow 1$;
for (i = 1; $i<n$; i++){
  if ($c_i\cap D=\varnothing$) $x\leftarrow x+1$;
  else $x\leftarrow x-1-2\left(\left|c_i\cap D\right|-1\right)$;
    $D_1=D_1\cup\left(c_i\cap D\right)$;
  if ($x=z$){
    $k\leftarrow i-z+1$;
  }
}
for(i = n; $i>k$; i--){
  if ($c_i\cap D=\varnothing$) $y\leftarrow y+1$;
  else $y\leftarrow y-1-2\left(\left|c_i\cap D\right|-1\right)$;
    $D_2=D_2\cup\left(c_i\cap D\right)$
  if($y=z$){
    $k'\leftarrow i+z-1$;
  }
}
Return k, D_1, D_2;

Theorem 1: *Let* $G=P_n[P_m]$ *for* $m>6$. *Then*

$$\gamma_{3,2,1}\left(P_n[P_m]\right)=\begin{cases}2, & 2\le n\le 3;\\ \frac{n}{2}+1, & n=6 \text{ or } n=10;\\ \left\lceil\frac{n}{2}\right\rceil, & \text{otherwise.}\end{cases}$$

Proof. For $2\le n\le 3$, let $D=\left\{\left(u_1,v_1\right),\left(u_1,v_2\right)\right\}$. Since $d\left((u_i,v_1),(u_1,v_1)\right)\le$ $d\left((u_i,v_3),(u_1,v_1)\right)=2$ and $d\left((u_i,v_1),(u_1,v_2)\right)=d\left((u_i,v_3),(u_1,v_2)\right)=1$ for $2\le i\le m$, we have $w_D\left((u_i,v_1)\right)\ge$ $w_D\left((u_i,v_3)\right)$ $=1+2=3$ for $2\le i\le m$. Similarly, for each $2\le i\le m$, vertex (u_i,v_2) is at distance 1 from vertex (u_1,v_1) and at distance no more than 2 from (u_1,v_2), so we have $w_D\left((u_i,v_2)\right)\ge 2+1=3$ for $2\le i\le m$. Hence D is a $D_{3,2,1}$-dominating set of $P_m[P_2]$. By Fact 1, $\gamma_{3,2,1}\left(P_m[P_2]\right)=2$. Next we show the upper bound of $\gamma_{3,2,1}\left(P_n[P_m]\right)$ for the rest cases:

Case 1: $n\ge 4$ and $n\not\equiv 2\,(\bmod 4)$. Consider $D=\left\{\left(u_1,v_{n-1}\right),\left(u_1,v_{n-2}\right)\right\}\cup$ $\left\{\left(u_1,v_j\right)\mid j=4i+2 \text{ or } 4i+3, \text{for } 0\le i\le\left\lfloor\frac{n}{4}\right\rfloor-1\right\}$. Since for all vertex $\left(u_k,v_j\right)\in V\text{-}D$, $N_1\left((u_k,v_j)\right)\cap D\ge 1$ and $N_2\left((u_k,v_j)\right)\cap D\ge 1$, we have $w_D\left(\left(u_i,v_j\right)\right)\ge 3$ for $1\le i\le m$ and $1\le j\le n$, which implies D is a $D_{3,2,1}$-dominating set of $P_n[P_m]$. Let $n=k\,(\bmod 4)$, then $\left|D\right|=2\left\lfloor\frac{n}{4}\right\rfloor+k$ $=\left\lceil\frac{n}{2}\right\rceil$. Hence $\gamma_{3,2,1}\left(P_n[P_m]\right)\le\left\lceil\frac{n}{2}\right\rceil$.

Case 2: $n=2\,(\bmod 4)$. This case may be divided into two subcases:

Subcase 2.1: $n=6,10$. The same D in case 1 will work in this case and $\left|D\right|=2\left\lfloor\frac{n}{4}\right\rfloor+2$ $=2\left\lceil\frac{n}{4}\right\rceil=\frac{n}{2}+1$.

Subcase 2.2: $n\ge 14$ and $n=2\,(\bmod 4)$. Consider $D=\left\{\left(u_1,v_j\right)\middle| j=4i+2 \text{ or } 4i+3 \text{ for } 0\le i\le\left\lfloor\frac{n}{4}\right\rfloor-3\right\}\cup\left\{(u_1,v_{n-1}),\left(u_1,v_{n-2}\right),\left(u_1,v_{n-5}\right),\left(u_1,v_{n-7}\right),\left(u_2,v_{n-7}\right)\right\}$ as shown in figure 4. Then D is a $D_{3,2,1}$-dominating set of $P_n[P_m]$ and $\left|D\right|=$ $2\left\lfloor\frac{n}{4}\right\rfloor+1$ $=\left\lceil\frac{n}{2}\right\rceil$, which implies $\gamma_{3,2,1}\left(P_n[P_m]\right)$ $\le\left\lceil\frac{n}{2}\right\rceil$.

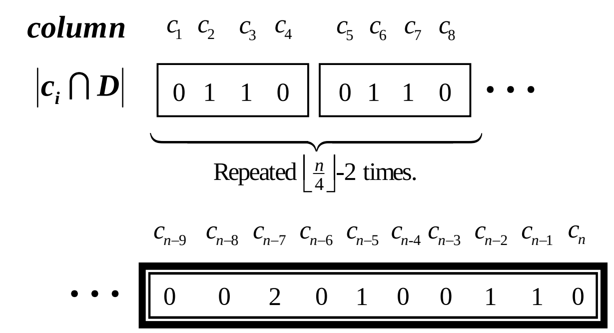

Figure 4: A pattern of $D_{3,2,1}$-dominating set for $P_n[P_m]$ where $n = 2\,(\mathrm{mod}\,4)$ and $n \geq 14$.

Next we show the lower bound of $\gamma_{3,2,1}(P_n[P_m])$. First we consider the case of $n \geq 4$, and $n \neq 6,10$. Suppose to the contrary that $\gamma_{3,2,1}(P_n[P_m]) \leq \lceil \frac{n}{2} \rceil - 1$. Then there exists a $D_{3,2,1}$-dominating set D such that $|D| = \frac{n}{2} - 1$ for even n and $|D| = \lfloor \frac{n}{2} \rfloor$ for odd n. Consider following two cases:

Case 1: n is even. By using algorithm 1 to "partition" the graph, we can get two consecutive columns c_k, c_{k+1} such that $(c_k \cup c_{k+1}) \cap D = \varnothing$, $\left|\bigcup_{i=1}^{k-1} c_i \cap D\right| = |D_1| = \frac{k-1}{2}$ and $\left|\bigcup_{i=k+2}^{n} c_i \cap D\right| = |D_2| = \frac{n-k-1}{2}$. Let D_1, D_2 be two sets produced by algorithm 1, notice that $|D_1| = \frac{k\text{-}1}{2}$ and $|D_2| = \frac{n\text{-}k\text{-}1}{2}$. Since D_1 can $D_{3,2,1}$-dominate the vertices from c_1 to $c_{k\text{-}1}$, and D_2 can $D_{3,2,1}$-dominate the vertices from c_{k+2} to c_n, to ensure that the vertices in c_k and c_{k+1} can also be $D_{3,2,1}$-dominated by $D_1 \cup D_2$, one of the following must be satisfied:

(1) $|c_{k\text{-}1} \cap D| = 1$ and $|c_{k+2} \cap D| = 1$

(2) $|c_{k\text{-}1} \cap D| = 1$, $c_{k+2} \cap D = \varnothing$ and $|c_{k+3} \cap D| = 2$

(3) $c_{k\text{-}1} \cap D = \varnothing$, $|c_{k\text{-}2} \cap D| = 2$ and $|c_{k+2} \cap D| = 1$

(4) $(c_{k\text{-}1} \cup c_{k+2}) \cap D = \varnothing$ and $|c_{k\text{-}2} \cap D| = |c_{k+3} \cap D| = 3$

The first three conditions have either $|c_{k\text{-}1} \cap D| = 1$ or $|c_{k+2} \cap D| = 1$. Without loss of generality, assume $|c_{k\text{-}1} \cap D| = 1$. By Lemma 2, $|D_1| > (k\text{-}1)/2$ which contradict to the fact that $|D_1| = (k\text{-}1)/2$, so it is impossible.

For (4), since $c_{k\text{-}1}$ to c_{k+2} are four consecutive columns without any vertex in $D_{3,2,1}$-domination set, which contradict to Lemma1, hence it is also impossible.

Case 2: n is odd. By using algorithm 1 to "partition" the graph, we can get one column c_k, such that $c_k \cap D = \varnothing$, $\left|\bigcup_{i=1}^{k-1} c_i \cap D\right| = |D_1| = \frac{k-1}{2}$ and $\left|\bigcup_{i=k+1}^{n} c_i \cap D\right| = |D_2| = \frac{n-k}{2}$. Since D_1 can $D_{3,2,1}$-dominate the vertices from c_1 to $c_{k\text{-}1}$, and D_2 can $D_{3,2,1}$-dominate the vertices from c_{k+1} to c_n, to ensure that the vertices in c_k can also be $D_{3,2,1}$-dominated by $D_1 \cup D_2$, one of the following must be satisfied:

(1) $|c_{k\text{-}1} \cap D| = 1$, $c_{k+1} \cap D = \varnothing$ and $|c_{k+2} \cap D| = 1$

(2) $|c_{k+1} \cap D| = 1$, $c_{k\text{-}1} \cap D = \varnothing$ and $|c_{k\text{-}2} \cap D| = 1$

(3) $(c_{k\text{-}1} \cup c_{k+1}) \cap D = \varnothing$, $|c_{k\text{-}2} \cap D| = 2$ and $|c_{k+2} \cap D| = 1$

(4) $(c_{k\text{-}1} \cup c_{k+1}) \cap D = \varnothing$, $|c_{k+2} \cap D| = 2$ and $|c_{k\text{-}2} \cap D| = 1$

(5) $(c_{k\text{-}1} \cup c_{k+1}) \cap D = \varnothing$ and $|c_{k\text{-}2} \cap D| = 3$

(6) $(c_{k\text{-}1} \cup c_{k+1}) \cap D = \varnothing$ and $|c_{k+2} \cap D| = 3$

Condition (1) and (2) have either $|c_{k\text{-}1} \cap D| = 1$ or $|c_{k+1} \cap D| = 1$, without loss of generality, assume $|c_{k\text{-}1} \cap D| = 1$. By Lemma 2, $|D_1| > (k\text{-}1)/2$ which contradict to the fact that $|D_1| = (k\text{-}1)/2$, so it is impossible.

Condition (3) to (6) have either $|c_{k\text{-}2} \cap D| \geq 2$ or $|c_{k+2} \cap D| \geq 2$, without loss of generality, assume $|c_{k+2} \cap D| \geq 2$. Consider following subcases.

Subcase 2.1: If $|c_{k+2} \cap D| = 2$, in order to fulfill the condition $w_D(v) \geq 3$ for all $v \in c_{k+2}$, we must have $|(c_{k+3} \cup c_{k+4}) \cap D| = x \geq 1$, assume that from

c_{k+1} through c_{k+2x+4}, there are no more than $x+2$ vertices in D. Since from c_{k+1} through c_{k+2x+4}, $\left|\left(N_1(v)\cup N_2(v)\right)\cap D_2\right|\le 1$ for all $v\in c_{k+2x+3}$, we must have $\left|c_{k+2x+5}\cap D\right|\ge 1$. By lemma 2, the subgraph from c_{k+2x+5} to c_n must have more than $\frac{n-(k+2x+5)+1}{2}$ vertices in $D_{3,2,1}$-dominating set, hence $\left|D_2\right|>\frac{n-(k+2x+5)+1}{2}+x+2=\frac{n-k}{2}$ which contradict to $\left|D_2\right|=\frac{n-k}{2}$, so it is impossible.

Subcase 2.2: If $\left|c_{k+2}\cap D\right|=x\ge 3$, assume that from c_{k+1} through c_{k+2x}, there are no more than x vertices in D. Since from c_{k+1} through c_{k+2x}, $\left(N_1(v)\cup N_2(v)\right)\cap D_2=\varnothing$ for all $v\in c_{k+2x}$, we must have $\left|c_{k+2x+1}\cap D\right|\ge 3$. By lemma 2, the subgraph from c_{k+2x+1} to c_n must have more than $\frac{n-(k+2x+1)+1}{2}$ vertices in $D_{3,2,1}$-dominating set, hence $\left|D_2\right|>\frac{n-(k+2x+1)+1}{2}+x=\frac{n-k}{2}$ which contradict to the fact that $\left|D_2\right|=\frac{n-k}{2}$, hence it is also impossible.

Next we consider the case of $n=6$. According to the definition of $D_{3,2,1}$-domination, if $w_D(v)\ge 3$ for all $v\in c_1$, then $\left|\bigcup_{i=1}^{3}c_i\cap D\right|\ge 2$. Similarly, if $w_D(v)\ge 3$ for all $v\in c_6$, then $\left|\bigcup_{i=4}^{6}c_i\cap D\right|\ge 2$. Hence $\gamma_{3,2,1}\left(P_6[P_m]\right)\ge 4=\frac{n}{2}+1$.

For the case of $n=10$, according to the definition of $D_{3,2,1}$-domination, if $w_D(v)\ge 3$ for all $v\in c_1$, then $\left|\bigcup_{i=1}^{3}c_i\cap D\right|\ge 2$. Similarly, if $w_D(v)\ge 3$ for all $v\in c_{10}$, then $\left|\bigcup_{i=8}^{10}c_i\cap D\right|\ge 2$. Consider following three cases:

Case 1: $\left|\{c_1,c_2,c_3\}\cap D\right|=3$ and $\left|\{c_8,c_9,c_{10}\}\cap D\right|=2$. In this case, $\forall v\in c_6$ $3\left|\{v\}\cap D\right|+2\left|N_1(v)\cap D\right|+\left|N_2(v)\cap D\right|<3$, therefore, $\gamma_{3,2,1}\left(P_{10}[P_m]\right)\ge 6$.

Case 2: $\left|\{c_1,c_2,c_3\}\cap D\right|=2$ and $\left|\{c_8,c_9,c_{10}\}\cap D\right|=3$. In this case, $\forall v\in c_5$ $3\left|\{v\}\cap D\right|+2\left|N_1(v)\cap D\right|+\left|N_2(v)\cap D\right|<3$, therefore, $\gamma_{3,2,1}\left(P_{10}[P_m]\right)\ge 6$.

Case 3: $\left|\{c_1,c_2,c_3\}\cap D\right|=2$ and $\left|\{c_8,c_9,c_{10}\}\cap D\right|=2$. In this case, according to the definition of $D_{3,2,1}$-domination, in order to have $w_D(v)\ge 3$ $\forall v\in c_1$, we must have $\left|c_3\cap D\right|\le 1$. Similarly, in order to have $w_D(v)\ge 3$ $\forall v\in c_{10}$, we must have $\left|c_8\cap D\right|\le 1$. Therefore, $\forall v\in\left(c_5\cup c_6\right)$ $3\left|\{v\}\cap D\right|+2\left|N_1(v)\cap D\right|+\left|N_2(v)\cap D\right|\le 1$. Since $\bigcap_{v\in c_5\cup c_6}N_1(v)=\varnothing$, for $w_D(v)\ge 3$ $\forall v\in\left(c_5\cup c_6\right)$, $\left|\{c_4,c_5,c_6,c_7\}\cap D\right|\ge 2$, therefore, $\gamma_{3,2,1}\left(P_{10}[P_m]\right)\ge 6$.

From all the cases above, we have the lower bound of $P_n[P_m]$ for $n\ge 4$ is the same as the upper bound. By sandwich theorem, we proved the theorem. ■

$P_n[C_m]$ changes each column of $P_n[P_m]$ from a path to a cycle, which does not changes the distance between columns. Hence Corollary can be obtained from Theory 1 directly.

Corollary 1 *Let* $G=P_n[C_m]$ *for* $m>6$. *Then*

$$\gamma_{3,2,1}\left(P_n[C_m]\right)=\begin{cases}2, & 2\le n\le 3;\\ \frac{n}{2}+1, & n=6 \text{ or } n=10;\\ \left\lceil\frac{n}{2}\right\rceil, & \text{otherwise.}\end{cases}$$

III. Conclusion

The $D_{3,2,1}$-domination is related to distance-two-domination, which has many applications in resource allocations. This paper established the $D_{3,2,1}$-domination number of the composition of a path with a path and a path with a cycle by giving detail proofs for each case. The author expect to study on the same problem for the composition of a cycle with a path and a cycle with a cycle in the near future.

REFERENCE

[1] R.B. Allan, R. Laskar and S.T. Hedetniemi, "A note on total domination," Discrete Mathematics, Vol. 49, pp.7–13, 1984.

[2] J. Cyman, M. Lemanska and J. Raczek, “Lower bound on the distance k-domination number of a tree,” Math. Slovaca, Vol.56(2), pp.235-243, 2006.

[3] P. Dankelmann, “Average distance and the domination number,” Discrete Applied Mathematics, Vol.80, pp.21-35, 1997.

[4] O. Favaron and D. Kratsch, “Ratios of domination parameters, Advances in Graph Theory,” Vishwa Int. Pub., pp.173-182, 1991.

[5] M. Fischermann and L. Volkmann, “Graphs having distance-n domination number half their order,” Discrete Applied Mathematics, Vol.120, pp.97–107, 2002.

[6] A. Hansberg, D Meierling and L Volkmann, “Distance domination and distance irredundance in graphs,” The Electronic Journal of Combinatorics, Vol.14, pp.#R35, 2007.

[7] J.H. Hattingh and M.A. Henning, “The ratio of the distance irredundance and domination numbers of a graph,” Journal of Graph Theory, Vol.18, pp.1–9, 1994.

[8] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, “Domination in Graphs: Advanced Topics,” Marcel Decker Inc, New York, 1998.

[9] T. Haynes, S.T. Hedetniemi and P.J. Slater, “Fundamentals of Domination in Graphs,” Marcel Decker Inc, New York, 1998.

[10] M.A. Henning, O.R. Oellermann and H.C. Swart, “Bounds on distance domination parameters,” Journal of Combinatorics, Information System Sciences, Vol.16, pp.11-18, 1991.

[11] M.A. Henning, O.R. Oellermann and H.C. Swart, “Relations between distance domination parameters,” Mathematica Pannonica, Vol.5, pp.69-79, 1994.

[12] F. R. Hsu, “Distance-two domination of graph,” Master Thesis, Department of Mathematic, National Central University, Taiwan, 2006.

[13] Shougui Li, “On connected k-domination numbers of graphs,” Discrete Mathematics, Vol.274, pp.303–310, 2004.

[14] D. Meierling and L. Volkmann, “A lower bound for the distance k-domination number of trees,” Results in Mathematics, Vol.47, pp.335-339, 2005.

[15] Fang Tian and Jun-Ming Xu, “A note on distance domination numbers of graphs,” Australasian Journal of Combinatorics, Vol.43, pp.181-190, 2009.

[16] D.B. West, “Introduction to Graph Theory (2nd ed.),” Prentice-Hall, NJ, 2001.

[17] K.Y. Zhen, “Distance-two domination of double-loop networks,” Master Thesis, Department of Mathematic, National Central University, Taiwan, 2008.