雙迴圈網路上的容錯號誌環嵌入
Fault Tolerant Token Ring Embedding in Double Loop Networks*

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摘要

一個雙迴圈網路 $G(n;s_1,s_2)$ 爲一有 n 個節點 $\{0,1,\ldots,n-1\}$ 與 $2n$ 個形式爲 $i \to i+s_1 \pmod n$ 和 $i \to i+s_2 \pmod n$ 的連線之有向圖形。一個雙迴圈網路將被稱做 *LFT*;若所有 $G(n;s_1,s_2)-e$ 均有漢米爾頓迴路,其中 e 爲任意邊線。而 $G(n;s_1,s_2)$ 將被稱做 *NFT*;若所有 $G(n;s_1,s_2)-v$ 均有漢米爾頓迴路,其中 v 爲任意節點。本文將提出 *LFT* 與 *NFT* 雙迴圈網路的充份、必要條件。

關鍵字:雙迴圈網路,漢米爾頓迴路

Abstract

A double loop network $G(n;s_1,s_2)$ is a digraph with n nodes $\{0,1,\ldots,n-1\}$ and $2n$ links of the form $i \to i+s_1 \pmod n$ and $i \to i+s_2 \pmod n$. A double loop network $G(n;s_1,s_2)$ is LFT if there is a hamiltonian cycle in every $G(n;s_1,s_2)-e$ where e is any link in the network. Similarly, a double loop network $G(n;s_1,s_2)$ is NFT if there is a hamiltonian cycle in every $G(n;s_1,s_2)-v$ where v is a node in the network. In this paper, we present necessary and sufficient conditions for LFT and NFT double loop networks, respectively.

Keywords: double loop network, hamiltonian cycle

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1 Introduction

The architecture of a local area network is always represented by a graph. We use graphs and networks interchangeably. In the design and implementation of local area networks, the ring topology has been used frequently due to its good properties such as simplicity, expandability, regularity and easiness of implementation. However, the ring network suffers drawbacks in low connectivity, low degree of fault tolerance and relatively large diameter. The performance of a ring network can be improved by adding links to it, which results in a so-called *loop network*. Among all loop networks, one may desire adding in a homogeneous way as few links to the ring structure as possible. Thus we are in particular interested in *double loop networks*.

A *double loop network*, denoted by $G(n;s_1,s_2)$, is a digraph with n nodes $\{0,1,\ldots,n-1\}$ and $2n$ links of the form $i \to i+s_1 \pmod n$ and $i \to i+s_2 \pmod n$, referred to as s_1-links and s_2-links, respectively. The undirected version of $G(n;,s_1,s_2)$ is given by $G(n;\pm s_1,\pm s_2)$ where each vertex i is adjacent to 4 vertices $i \pm s_1$ and $i \pm s_2$. Double loop networks are extensions of ring networks and widely used in the design and implementation of local area networks.

In this paper, we consider only directed version of double loop networks, i.e., $G(n;s_1,s_2)$. To construct an n-vertex double loop network, the choice of s_1 and s_2 is a vital issue. In literature, different values of s_1 and s_2 are chosen to achieve some desired properties. For example, Wong and Coppersmith [12] showed that choosing $s_1 = 1$ and s_2 around $\sqrt{n}$ yields a small diameter, approximate to $2\sqrt{n}$, and a small average distance, approximate to $\sqrt{n}$. Readers can refer to [1, 8] for a survey on double loop networks. In addition to small diameter and small average distance, fault tolerance is an important concern in the design of interconnection networks.

We say that a graph G is *hamiltonian* if there is a hamiltonian cycle in G. We say that a double loop network $G(n; s_1, s_2)$ is LFT (1-link fault-tolerant) if every $G(n; s_1, s_2) - e$ is hamiltonian where e is a link in $G(n; s_1, s_2)$. Similarly, a double loop network $G(n; s_1, s_2)$ is NFT (1-node fault-tolerant) if every $G(n; s_1, s_2) - v$ is hamiltonian where v is a node in $G(n; s_1, s_2)$. It can be easily verified that for $i = 1, 2$, all of the s_i links form a hamiltonian cycle if $gcd(n, s_i) = 1$. Thus $G(n; s_1, s_2)$ is LFT if $gcd(s_1, n) = 1$ and $gcd(s_2, n) = 1$. However, the converse is not necessarily true. For example, $G(12; 5, 2)$ is LFT but $gcd(12, 2) \neq 1$. Furthermore, $G(n; 1, 2)$ is NFT, but $G(n; 1, 3)$ is not NFT when n is even.

In this paper, we present necessary and sufficient conditions for LFT and NFT double loop networks, respectively. In the design of a local area network using double loop network architecture, we choose the one which has a small diameter and is both LFT and/or NFT.

Throughout this paper, we adopt the following notation. We define s as $s = s_1 - s_2 \pmod{n}$ and $d = gcd(n, s)$. Let C be a cycle in $G(n; s_1, s_2)$. We write for edge $e \in C$ to mean that e is an edge in C.

2 LFT double loop networks

When $s_1 = s_2 \pmod{n}$, it is obvious that $G(n; s_1, s_2)$ is LFT if and only if $gcd(n, s_1) = 1$. Thus we consider only the case $s_1 \neq s_2 \pmod{n}$ in this section. For $0 \leq i < d$, let $T_i = \{j \mid 0 \leq j < n$ and $j = i \pmod{d}\}$. Obviously, all T_i with $0 \leq i < d$ form a partition of $\{0, 1, \ldots, n-1\}$, and each T_i contains n/d elements. For any integer $0 \leq m < n$, we use $[m]$ to denote the integer i such that $m \in T_i$. Obviously, $[m] = m \pmod{d}$. Let C be a hamiltonian cycle of $G(n; s_1, s_2)$. For any vertex i in $G(n; s_1, s_2)$ we define a function f_C as follows:

$$f_C(i) = \begin{cases} s_1 & \text{if } (i, i+s_1 \pmod{n}) \in C, \\ s_2 & \text{otherwise.} \end{cases}$$

Lemma 1 *Let C be any hamiltonian cycle in $G(n; s_1, s_2)$. For any two vertices j and k in T_i with $0 \leq i < d$, we have $f_C(j) = f_C(k)$.*

Proof Without loss of generality, we assume that $f_C(i) = s_1$. Let $m = i + s \pmod{n}$. Suppose that $f_C(m) = s_2$. It follows that $(i, i+s_1 \pmod{n}) \in C$ and $(m, m+s_2 \pmod{n}) \in C$. Equivalently, both $(i, i+s_1 \pmod{n})$ and $(m, i+s_1 \pmod{n})$ are in C. There are two links in C entering the vertex $i + s_1 \pmod{n}$, which is contradictory to the fact that C is a hamiltonian cycle. Thus, $f_C(m) = s_1$. Recursively, any vertex $k = i+ts \pmod{n}$ for some integer t also satisfies $f_C(k) = s_1$. On the other hand, the set $\{i + ts \pmod{n} \mid t \geq 0$ and integer$\}$ constitutes T_i for every $0 \leq i < d$. Therefore the lemma is proved. □

For any hamiltonian cycle C, we define a function g_C from $\{0, 1, \ldots, n-1\}$ into $\{T_i \mid 0 \leq i < d\}$ as follows:

$$g_C(m) = T_{[m+f_C(m)]}.$$

In other words, $g_C(m)$ denotes the set T_i to which the vertex next to m in C belongs. It follows from Lemma 1 that

$$g_C(T_i) = T_{[i+f_C(i)]} \quad \text{for } 0 \leq i < d, \text{ and}$$
$$g_C(\{0, 1, \ldots, d-1\}) = \{T_i \mid \forall\, 0 \leq i < d\}.$$

Let H be a digraph with the vertex set $\{T_i \mid \forall 0 \leq i < d\}$ and link set $\{(T_i, g_C(T_i)) \mid \forall 0 \leq i < d\}$. Obviously, H is a directed cycle with d elements.

Lemma 2 *If $G(n; s_1, s_2)$ is hamiltonian, then $gcd(d, s_1) = gcd(d, s_2) = 1$.*

Proof. Let $r = gcd(d, s_1)$. Obviously, r is also $gcd(d, s_2)$. Suppose that $G(n; s_1, s_2)$ has a hamiltonian cycle C and $r > 1$. Let H be defined as above. It follows that H is a connected directed cycle. However, g_C maps the set $\{T_i \mid i$ is a multiple of $r\}$ onto itself and thus, H is disconnected which is a contradiction. Consequently, the lemma is proved. □

Lemma 3 *Let $G(n; s_1, s_2)$ be a double loop network with $gcd(d, s_1) = 1$. Let α, β, γ, and δ be non-negative integers satisfying $\alpha \leq \gamma$, $\beta \leq \delta$, and $\alpha + \beta < \gamma + \delta < d$. Then $T_{[\alpha s_1 + \beta s_2]} \neq T_{[\gamma s_1 + \delta s_2]}$.*

Proof. Let $G(n; s_1, s_2)$ be a double loop network with $gcd(d, s_1) = 1$, i.e., $gcd(d, s_2) = 1$ as well. Let α, β, γ and δ be nonnegative integers satisfying $\alpha \leq \gamma$, $\beta \leq \delta$, $\alpha + \beta < \gamma + \delta < d$, and $T_{[\alpha s_1 + \beta s_2]} = T_{[\gamma s_1 + \delta s_2]}$. Since $T_{[\alpha s_1 + \beta s_2]} = T_{[\gamma s_1 + \delta s_2]}$, it follows that $\alpha s_1 + \beta s_2 \pmod{d} = \gamma s_1 + \delta s_2 \pmod{d}$. Therefore, $(\alpha - \gamma)s_1 + (\beta - \delta)s_2 = 0 \pmod{d}$ and equivalently, $(\alpha - \gamma)(s_2 + s) + (\beta - \delta)s_2 = 0 \pmod{d}$. Since $s = 0 \pmod{d}$, it follows that $(\alpha + \beta - \gamma -$

$\delta)s_2 = 0 \pmod d$. Since $gcd(d, s_2) = 1$, it follows that $\alpha+\beta-\gamma-\delta = 0 \pmod d$ which contradicts $0 \leq \alpha+\beta < \gamma+\delta < d$. Hence $T_{[\alpha s_1+\beta s_2]} \neq T_{[\gamma s_1+\delta s_2]}$, and the lemma is proved. □

Lemma 4 *$G(n; s_1, s_2)$ is hamiltonian if and only if (i) $gcd(d, s_1) = 1$, and (ii) there exists an integer k, $0 \leq k \leq d$, satisfying $gcd((ks_1+(d-k)s_2)/d, n/d) = 1$.*

Proof. To prove the necessity, assume that $G(n; s_1, s_2)$ is hamiltonian. It follows from Lemma 2 that $gcd(d, s_1) = 1$. Let $C = \langle x_0 = 0, x_1, \ldots, x_{n-1}, x_0\rangle$ be a hamiltonian cycle. We define a sequence of ordered pairs of nonnegative integers $\{(\alpha_i, \beta_i) \mid 0 \leq i \leq n-1\}$ as follows:

$(\alpha_0, \beta_0) = (0, 0)$

$$(\alpha_i, \beta_i) = \begin{cases} (\alpha_{i-1}, \beta_{i-1}) + (1, 0) & \text{if } x_i = x_{i-1} + s_1 \pmod n, \\ (\alpha_{i-1}, \beta_{i-1}) + (0, 1) & \text{otherwise.} \end{cases}$$

It follows from the definition of (α_i, β_i) that $i = \alpha_i + \beta_i$ and $x_i = \alpha_i s_1 + \beta_i s_2 \pmod n$. Since any two pairs (α_i, β_i), (α_j, β_j) with $0 \leq i < j < d$ satisfy $0 \leq \alpha_i + \beta_i < \alpha_j + \beta_j$, it follows from Lemma 3 that $T_{[x_i]} \neq T_{[x_j]}$. By the pigeon-hole principle, $\{T_i \mid 0 \leq i < d\} = \{T_{[x_i]} \mid 0 \leq i < d\}$. Since $s = 0 \pmod d$ and $x_d = ds_1 - \beta_d s \pmod d$, we have $T_{[x_d]} = T_{[x_0]}$. Furthermore, it follows from Lemma 1 that $T_{[x_{d+1}]} = T_{[x_1]}$. Recursively, we have $T_{[x_j]} = T_{[x_{j \pmod d}]}$. Let $k = \alpha_d = \mid \{i \mid f_C(i) = s_1,\ 0 \leq i < d\} \mid$. It follows from Lemma 1 that $x_{td} = t(ks_1 + (d-k)s_2) \pmod n$ for all $1 \leq t < n/d$. Suppose that $gcd(n/d, x_d/d) = r > 1$. Then $n/d = ar$, $x_d/d = br$ for some integers a, b with $gcd(a, b) = 1$. Obviously, $a < n/d$. We note that $ax_d/d = abr = bn/d$ is a multiple of n/d. That is, ax_d is a multiple of n which contradicts $x_{td} = tx_d \neq 0 \pmod n$ for all $1 \leq t < n/d$. Thus $gcd(x_d/d, n/d) = gcd((ks_1 + (d-k)s_2)/d, n/d) = 1$ for $0 \leq k \leq d$.

On the other hand, suppose that $gcd(d, s_1) = 1$ and $gcd((ks_1+(d-k)s_2)/d, n/d) = 1$ for an integer k where $0 \leq k \leq d$. We construct a sequence $D = \langle y_0 = 0, y_1, \ldots, y_{n-1}\rangle$ as follows:

$y_0 = 0$

$$y_i = \begin{cases} y_{i-1} + s_1 \pmod n & \text{if } 1 \leq i \pmod d \leq k, \\ y_{i-1} + s_2 \pmod n & \text{otherwise.} \end{cases}$$

In other words, $y_j = y_{j \pmod d} + \lfloor \frac{j}{d} \rfloor (ks_1 + (d-k)s_2) \pmod n$. Obviously, (y_{i-1}, y_i) is a link of G. In order to prove that D forms a hamiltonian cycle in $G(n; s_1, s_2)$, we are required to show $y_i \neq y_j$ for all $0 \leq i < j < n$. Since $gcd(d, s_1) = 1$, it follows from Lemma 3 that $\{T_{[y_i]} \mid 0 \leq i < d\} = \{T_i \mid 0 \leq i < d\}$. Since $ks_1 + (d-k)s_2 = 0 \pmod d$, it follows that $y_j \in T_{[y_{j \pmod d}]}$. Hence $y_i \neq y_j$ if $i \neq j \pmod d$. Since $gcd((ks_1+(d-k)s_2)/d, n/d) = 1$, it follows that $y_{td} \neq 0 \pmod n$ for all $1 \leq t < n/d$. Then we have $y_i \neq y_j$ for all $i = j \pmod d$ and $0 \leq i < j < n$. Hence the theorem is proved. □

Using the proof of the above lemma, we can easily obtian the following corollaries.

Corollary 1 *$G(n; s_1, s_2)$ contains a hamiltonian cycle with at least one s_1-link and one s_2-link if and only if (i) $gcd(d, s_1) = 1$, and (ii) there exists an integer k, $1 \leq k < d$, satisfying $gcd((ks_1 + (d-k)s_2)/d, n/d) = 1$.*

Theorem 1 *$G(n; s_1, s_2)$ is LFT if and only if at least one of the following statements holds:*

(a) $gcd(d, s_1) = 1$, and there exists an integer k with $1 \leq k < d$ such that $gcd((ks_1 + (d-k)s_2)/d, n/d) = 1$;

(b) $gcd(n, s_i) = 1$ for $i = 1$ and 2.

Proof. To prove the necessity, suppose $G(n; s_1, s_2)$ is *LFT*. $G(n; s_1, s_2)$ contains a hamiltonian cycle with at least one s_1-link and one s_2-link or two hamiltonian cycles using only s_1-links and s_2-links, respectively. It follows from Corollary 1 that Statement (a) holds in the former case. In the latter case, it is trivial that Statement (b) holds.

On the other hand, Statement (a) implies the existence of a hamiltonian cycle with at one s_1-link and one s_2-link. Therefore, $G(n; s_1, s_2)$ is *LFT* since $G(n; s_1, s_2)$ is node-symmetric. When Statement (b) holds, it is trivial that $G(n; s_1, s_2)$ is *LFT*. Hence the theorem follows. □

3 NFT double loop networks

When $s_1 = s_2 \pmod n$, $G(n; s_1, s_1)$ cannot be *NFT*. In the following discussion, we assume $s_1 \neq s_2 \pmod n$. In $G(n; s_1, s_2)$, we construct a sequence $N_1 = \{a_0^1, a_1^1, \ldots\}$ as follows:

$a_0^1 = n - s_2 \pmod n$,

$a_i^1 = n - s_2 + is \pmod n$ if $is \neq 0, s_2 \pmod n$.

The sequence terminates when $is = 0 \pmod n$ or $s_2 \pmod n$. It is obvious that $n + \frac{n}{d}s = 0 \pmod n$ or $is = s_2 \pmod n$ for some integer i. Thus the sequence N_1 is finite. Similarly, construct a sequence $N_2 = \{a_0^2, a_1^2, \ldots\}$ as follows:

$a_0^2 = n - s_1 \pmod n$,

$a_i^2 = n - s_1 - is \pmod n$ if $is \neq 0, -s_1 \pmod n$.

The sequence terminates when $is = 0 \pmod n$ or $is = -s_1 \pmod n$. N_2 is also finite. Note that the vertex 0 is not in either N_1 or N_2, i.e., $0 \notin N_1 \cup N_2$.

Lemma 5 *Let $G(n; s_1, s_2)$ be a double loop network with $gcd(n, s) = 1$. Let $|N_1| = x$ and $|N_2| = y$. Then x and y are the smallest positive integers satisfying $xs = s_2 \pmod n$ and $-ys = s_1 \pmod n$, respectively. Moreover, $x + y = n - 1$, $N_1 \cap N_2 = \emptyset$, and $N_1 \cup N_2 = \{1, 2, \ldots, n-1\}$.*

Proof. Since $gcd(n, s) = 1$, N_1 and N_2 terminate when $xs = s_2 \pmod n$ and $ys = -s_1 \pmod n$ are satisfied. Therefore, x is the smallest positive integer with $xs = s_2 \pmod n$ and y is the smallest positive integer with $-ys = s_1 \pmod n$. Obviously, $1 \le x < n$ and $1 \le y < n$. Since $(x + y)s = s_2 - s_1 \pmod n$ and $gcd(n, s) = 1$, we have $x + y = -1 \pmod n$. Thus, $x + y = n - 1$ follows from $2 \le x + y \le 2n - 2$.

Suppose $N_1 \cap N_2 \ne \emptyset$. Let $a^1_i = a^2_j$, where $0 \le i \le x-1, 0 \le j \le y-1$. It follows that $n - s_2 + is = n - s_1 - js \pmod n$ and $(1+i+j)s = 0 \pmod n$. Since $gcd(s, n) = 1$, it follows that $1 + i + j = 0 \pmod n$. It implies $i + j = n - 1$ which contradicts $i + j \le x + y - 2$ and $x + y = n - 1$. Thus $N_1 \cap N_2 = \emptyset$. Since $x + y = n - 1$, $N_1 \cap N_2 = \emptyset$ and $0 \notin N_1 \cup N_2$, it follows that $N_1 \cup N_2 = \{1, 2, \ldots, n-1\}$. The lemma follows. □

Let $G(n; s_1, s_2)$ be an NFT double loop network. There exists a hamiltonian cycle in $G(n; s_1, s_2) - v$ for every vertex $v \in \{0, 1, \ldots, n-1\}$. Since $G(n; s_1, s_2)$ is node-symmetric, we can assume without loss of generality that $v = 0$ in the following analysis. Let C be a hamiltonian cycle in $G(n; s_1, s_2) - 0$. For any vertex $i \in \{1, 2, \ldots, n-1\}$, define a function h_C as follows:

$$h_C(i) = \begin{cases} s_1 & \text{if } (i, i + s_1 \pmod n) \in C, \\ s_2 & \text{otherwise.} \end{cases}$$

Lemma 6 *Let $G(n; s_1, s_2)$ be an NFT double loop network, and C be a hamiltonian cycle in $G(n; s_1, s_2) - 0$. Then*

[i] $h_C(i) = s_1$ for all vertices $i \in N_1$,

[ii] $h_C(i) = s_2$ for all veritces $i \in N_2$.

Proof. Since $a^1_0 + s_2 = 0 \pmod n$, it follows that $(a^1_0, a^1_0 + s_2 \pmod n) \notin C$. Thus, $(a^1_0, a^1_0 + s_1 \pmod n) \in C$ and $h_C(a^1_0) = s_1$. Assume that $h_C(a^1_{k-1}) = s_1$ where $k < |N_1|$, i.e., $(a^1_{k-1}, a^1_{k-1} + s_1 \pmod n) \in C$. Note that $a^1_k + s_2 = (a^1_{k-1} + s) + s_2 = a^1_{k-1} + s_1 \pmod n$. Thus, $(a^1_k, a^1_k + s_2 \pmod n) \notin C$. Consequently, $(a^1_k, a^1_k + s_1 \pmod n) \in C$ and $h_C(a^1_k) = s_1$. Therefore, $h_C(i) = s_1$ for all $i \in N_1$. Similarly, we can prove $h_C(i) = s_2$ for all vertices $i \in N_2$. □

Lemma 7 *If $G(n; s_1, s_2)$ is an NFT double loop network, then $gcd(n, s) = 1$.*

Proof. Suppose to the contrary that $gcd(n, s) = d > 1$. We first consider the case $d \mid s_1$. It follows that $d \mid s_2$ also holds. Then all of the vertices in $\{i \mid d \mid i\}$ are adjacent to vertices in the same set. Thus $G(n; s_1, s_2)$ is disconnected and furthermore, $G(n; s_1, s_2)$ is not NFT which is a contradiction. Now, we consider the case $d \nmid s_1$. It follows that $d \nmid s_2$. Since s but not s_2 is a multiple of d, it follows that $is = s_2 \pmod n$ has no solution. We also note that $is = 0 \pmod n$ if and only if $i = 0 \pmod{\frac{n}{d}}$. Thus, $n - s_1 - (\frac{n}{d} - 1)s \pmod n$ is in N_2. However, $n - s_1 - (\frac{n}{d} - 1)s = n - s_2 \pmod n$ is also an element in N_1. It follows from Lemma 6 that we have $s_1 = h_C(n - s_2) = s_2$ which is contradictory to $s_1 \ne s_2$. Hence $gcd(n, s) = 1$. □

Theorem 2 *$G(n; s_1, s_2)$ is NFT if and only if the following conditions hold:*

(a) $s_1 \ne s_2$,

(b) $gcd(n, s) = 1$,

(c) $gcd(x, n-1) = 1$ where x is the smallest positive integer satisfying $xs = s_2 \pmod n$.

Proof. To prove the necessity, let $G(n; s_1, s_2)$ be NFT. It follows from Lemma 7 that $gcd(n, s) = 1$. It follows from Lemma 5 that we can construct a sequence $B = \{b_0, b_1, \ldots, b_{n-2}\}$ as follows:

$$b_i = \begin{cases} a^1_i & = n - s_2 + is \pmod n \ \text{if } 0 \le i \le x - 1, \\ a^2_{n-i-2} & = n - s_1 - (n - 2 - i)s \pmod n \\ & \text{if } x \le i \le n - 2, \end{cases}$$

where x is the smallest integer satisfying $xs = s_2 \pmod n$.

Let C be a hamiltonian cycle in $G(n; s_1, s_2) - 0$. Let y be the smallest positive integer satisfying $-ys = s_1 \pmod n$. We claim that for all $0 \le i < n - 1$, $b_i + h_C(b_i) = b_j \pmod n$ where $j = i + x \pmod{n-1}$. To prove the claim, we assume without loss of generality that $x \le y$ for ease of exposition.

First consider $0 \le i \le x - 1$. It follows from Lemma 6 that $h_C(b_i) = s_1$. Thus

$$b_i + h_C(b_i) = n + (i+1)s = n - s_1 - ys + (i+1)s$$
$$= a^2_{y-i-1} = b_j \pmod n,$$

where $j = i + x \pmod{n-1}$.

Consider $x \le i \le n-2-x$. It follows from Lemma 6 that $h_C(b_i) = s_2$. Thus, we have

$$b_i + h_C(b_i) = n - (n - 1 - i)s$$
$$= n - s_1 - (n - 2 - i - x)s$$
$$= a^2_{n-2-i-x} = b_j \pmod n,$$

where $j = i + x \pmod{n-1}$.

For $n-1-x \leq i < n-1$, we have $b_i = a^2_{n-2-i} = n - s_1 - (n-2-i)s \pmod{n}$ and $h_C(b_i) = s_2$. Thus,

$$b_i + h_C(b_i) = n - (n-1-i)s$$
$$= n - s_2 + (i + x - n + 1)s$$
$$= a^1_{i+x-(n-1)} = b_j \pmod{n},$$

where $j = i + x \pmod{n-1}$. The case $x > y$ can be similarly treated.

From the above discussion, C is uniquely determined by the sequence $D = \{b_{jx \pmod{n-1}} \mid 0 \leq j < n-1\}$. In this case, $\{jx \pmod{n-1} \mid 0 \leq j < n-1\} = \{0, 1, \ldots, n-2\}$. Therefore, $gcd(x, n-1) = 1$.

On the other hand, assume that $gcd(n,s) = 1$ and $gcd(x, n-1) = 1$, where x is the smallest positive integer satisfying $xs = s_2 \pmod{n}$. To prove $G(n; s_1, s_2)$ being NFT, we need to construct a hamiltonian cycle in $G(n; s_1, s_2) - 0$. Let $N_1 = \{a^1_0, a^1_1, \ldots, a^1_{x-1}\}$ and $N_2 = \{a^2_0, a^2_1, \ldots, a^2_{n-x-2}\}$. Since $gcd(n,s) = 1$, it follows from Lemma 5 that $N_1 \cup N_2 = \{1, 2, \ldots, n-1\}$. We define a new sequence $B = \{b_0, b_1, \ldots, b_{n-2}\}$ by setting $b_i = a^1_i$ if $0 \leq i \leq x-1$ and $b_i = a^2_{n-i-2}$ if $x \leq i \leq n-2$. We claim that the sequence $\{b_{\langle 0\rangle}, b_{\langle x\rangle}, b_{\langle 2x\rangle}, \ldots, b_{\langle (n-2)x\rangle}, b_{\langle 0\rangle}\}$ forms a hamiltonian cycle in $G(n; s_1, s_2) - 0$, where $\langle a\rangle = a \pmod{n-1}$. For ease of exposition, we assume without loss of generality that $x \leq y$. Then for $0 \leq i < x$,

$$b_{\langle i\rangle} = a^1_i = n - s_2 + is \pmod{n},$$
$$b_{\langle i+x\rangle} = a^2_{y-i-1} = n - s_1 - (y-1-i)s$$
$$= n + (i+1)s \pmod{n}.$$

Thus, $b_{\langle i+x\rangle} - b_{\langle i\rangle} = s_1 \pmod{n}$ for $0 \leq i < x$.

For $x \leq i < n-1-x$,

$$b_{\langle i\rangle} = a^2_{n-2-i} = n - s_1 - (n-2-i)s \pmod{n},$$
$$b_{\langle i+x\rangle} = a^2_{n-2-i-x}$$
$$= n - s_1 - (n-2-i-x)s$$
$$= n - (n-1-i)s \pmod{n}.$$

Thus, $b_{\langle i+x\rangle} - b_{\langle i\rangle} = s_2 \pmod{n}$ for $x \leq i < n-1-x$. For $n-1-x \leq i < n-1$,

$$b_{\langle i\rangle} = a^2_{n-2-i} = n - s_1 - (n-2-i)s \pmod{n},$$
$$b_{\langle i+x\rangle} = a^1_{i+x-(n-1)} = n - s_2 + (i+x-n+1)s$$
$$= n - (n-1-i)s \pmod{n}.$$

Thus, $b_{\langle i+x\rangle} - b_{\langle i\rangle} = s_2 \pmod{n}$ for $n-1-x \leq i < n-1$. Therefore, $(b_{\langle jx\rangle}, b_{\langle (j+1)x\rangle})$ is a link for all $0 \leq i < n-1$. Since $gcd(x, n-1) = 1$, we have $\{\langle jx\rangle \mid 0 \leq j < n-1\} = \{0, 1, \ldots, n-2\}$. Thus $\{b_{\langle jx\rangle} \mid 0 \leq j < n-1\} = \{1, 2, \ldots, n-1\}$. Therefore, the sequence $\{b_{\langle 0\rangle}, b_{\langle x\rangle}, b_{\langle 2x\rangle}, \ldots, b_{\langle (n-2)x\rangle}, b_{\langle 0\rangle}\}$ forms a hamiltonian cycle in $G(n; s_1, s_2) - 0$. Hence, the theorem follows. □

4 Discussion

For any positive integer n, the number of all possible (s_1, s_2) pairs is $C(n, 2)$. In this paper, we present necessary and sufficient conditions for LFT and NFT double loop networks. Using these conditions, we write a program to calculate the number of (s_1, s_2) pairs of $G(n; s_1, s_2)$ to achieve the desire properties of hamiltonicity, LFT, NFT, and $LNFT$ which means both NFT and LFT. We list results for all $51 \leq n \leq 70$ in Table 1.

Let n be even, and let s_1 and s_2 have the same parity. Then $gcd(n,s) \neq 1$. Therefore, when n is even and $s_1 - s_2 = 0 \pmod{2}$, $G(n; s_1, s_2)$ is not NFT following from Theorem 2. Let n be even, and let s_1 and s_2 have different parity. Without loss of generality, we assume that s_1 is even and s_2 is odd. Consider $G(n; s_1, s_2)$ is NFT. It follows from Theorem 2 that $d = gcd(n,s) = 1$. In this case, $gcd(n, s_1) \neq 1$ and there is no integer k with $1 \leq k < d$, not to mention satisfying $gcd((ks_1 + (d-k)s_2)/d, n/d) = 1$. Thus, when $G(n; s_1, s_2)$ is NFT, $G(n; s_1, s_2)$ is not LFT. Therefore, when n is even, there is no $LNFT$ double loop network, which can also be observed from Table 1.

It is also observed that when n is odd, the number of double loop networks $G(n; s_1, s_2)$ that are $LNFT$ is not small. It is possible to choose a double loop network that is $LNFT$. Moreover, when n is prime, every NFT double loop network $G(n; s_1, s_2)$ is also LFT.

In addition to fault tolerance, diameter is another performance measure of interconnection networks. Let $d(n; s_1, s_2)$ denote the diameter of $G(n; s_1, s_2)$. Let $d(n)$ denote the minimum diameter among all double loop networks having n vertices. Among those $G(n; s_1, s_2)$ achieving $d(n)$, is there always one also NFT or LFT or $LNFT$? The answer is no. For example, with the aid of a computer program we know that $d(12) = 5$, in particular, $d(12; 1, 8) = 5$. However, $G(12; 1, 8)$ is NFT but not LFT. There is no $G(12; s_1, s_2)$ which is $LNFT$. Let $n = 21$. We know that $d(21) = 6$. In particular $d(21; 1, 9) = d(21; 1, 13) = 6$. However, $G(21; 1, 13)$ is LFT but not NFT, whereas $G(21; 1, 9)$ is neither LFT nor NFT. We also find that there is no $G(21; s_1, s_2)$ with diameter 6 which is also $LNFT$. Consider a prime number $n = 59$. We find that $d(59; 1, 27) = d(59; 1, 35) = 12$ and that $G(59; 1, 27)$ and $G(59; 1, 35)$ are $LNFT$. Suppose that p is a prime number such that $p \neq 2 \pmod 3$. Consider a double loop network $H = G(p^2; p+2, 1)$. It follows from Theorem 1 that H is LFT. Let x be the smallest positive integer satisfying $x(p+1) = 1 \pmod{p^2}$. Note that $(p+1)(p-1) = -1 \pmod{p^2}$ implies $x = p^2 - p + 1$. It is observed that $gcd(p^2-p+1, p^2-1) = gcd(p^2-p+1, p-2) = gcd(p+1, p-2) = gcd(3, p-2) = 1$. Therefore, we obtain $gcd(p^2-p+1, p^2-1) = gcd(p^2, p+1) = 1$. It follows from Theorem 2 that H is NFT. Note that $p+2$ is close to $\sqrt{p^2}$. Applying the result of Wong and Coppersmith [12], both the diameter and the average distance of H are $O(p)$. In other words, H is $LNFT$ and has a small diameter.

n	$C(n,2)$	*Hamiltonian*	*LFT*	*NFT*	*LNFT*
50	1225	790	410	420	0
51	1275	1104	592	320	128
52	1326	948	372	384	0
53	1378	1378	1378	624	624
54	1431	801	441	468	0
55	1485	1380	900	360	200
56	1540	1044	444	480	0
57	1596	1386	738	432	108
58	1653	1218	462	504	0
59	1711	1711	1711	812	812
60	1770	824	552	464	0
61	1830	1830	1830	480	480
62	1891	1395	525	900	0
63	1953	1602	882	540	108
64	2016	1520	528	576	0
65	2080	1944	1272	768	432
66	2145	1110	730	480	0
67	2211	2211	2211	660	660
68	2278	1648	624	1056	0
69	2346	2046	1078	704	264
70	2415	1380	852	528	0

Table 1. A comparison on the numbers of hamiltonian, LFT, NFT and LNFT double loop netwworks.

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