Hamiltonian-Laceability in Star Graphs with Conditional Edge Faults

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ABSTRACT

The star graph S_n possess many nice topological properties. It is one of the most versatile and efficient interconnection networks (networks for short) so far discovered for parallel computation. Edge fault tolerance is an important issue for a network since the edges in the network may fail sometimes. It is known that S_n is a bipartite graph. In this paper, we show that for any S_n ($n \geq 4$) with $\leq 2n - 7$ edge faults in which each node is incident to at least two healthy edges is strongly Hamiltonian laceable. It is also shown that our result is optimal.

1: INTRODUCTIONS

The star graph [1], which is a Cayley graph, has been recognized as an attractive to the hypercube. It possesses many nice topological properties, such as recursiveness, symmetry, maximal fault tolerance, sublogarithmic degree and diameter, and strong resilience, which are all desirable when we are building an interconnection topology for a parallel and distributed system. The star graph can embed rings [16], grids [10], trees [3], and hypercubes [15]. Many efficient communication algorithms for shortest-path routing [16], multiple-path routing [4], broadcasting [14], and scattering [5] were proposed.

Linear arrays and rings, which are two of the most fundamental networks for parallel and distributed computation, are suitable for designing simple algorithms with low communication costs. Numerous efficient algorithms designed on linear arrays and rings for solving various algebraic problems and graph problems can be found in [2], [11]. Linear arrays and rings can also be used as control/data flow structures for distributed computation in arbitrary networks. These applications motivate the embedding of paths and cycles in networks.

Suppose that W is an interconnection network. A path (or cycle) in W is called a *Hamiltonian path* (or *Hamiltonian cycle*) if it contains every node of W exactly once. W is called *Hamiltonian* if there is a Hamiltonian cycle in W. A Hamiltonian network can embed a longest cycle with dilation 1, congestion 1, load 1, and expansion 1. Since processor or link faults may develop in real world networks, it is important to consider faulty networks. The problems of diameter, routing, gossiping, multicasting, broadcasting, and embedding have been solved on various faulty networks. This study considers the embedding problem on a faulty star graph. Previously related work can be found in [7], [8], [9], [13].

Let S_n denote an n-dimensional star graph. In this study, we show that for any S_n ($n \geq 4$) with $\leq 2n - 7$ edge faults in which each node is incident to at least two healthy edges, we can obtain a fault-free Hamiltonian path between two arbitrary nodes in different partite sets and a fault-free path of length $n! - 2$ between two arbitrary nodes in the same partite set. In the next section, necessary definitions and notations are first introduced. Then, the embedding is shown in Section 3. Finally, this paper concludes with some remarks in Section 4.

2: PRELIMINARIES

For convenience of discussing, a network topology is represented by a simple undirected graph, which is loopless and without multiple edges. For the graph definition and notation, we follow [17]. A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices called its endpoints. When vertices u and v are the endpoints of an edge, they are adjacent and are neighbors. Since a graph is simple undirected, we use (u, v) to denote the edge that connects vertex u and v. For a node u, $N(u)$ denotes the neighborhood of u, which is the set $\{v \mid (u, v) \in E\}$. A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A path is denoted by a sequence of adjacent vertices $<v_0, v_1, \ldots, v_k>$, in which $v_0, v_1, \ldots, v_k$ are distinct except that possibly $v_0 = v_k$. The length of the path is k. For ease of description, it is abbreviated to a v_0-v_k path, and we may use P or $<v_0, P, v_k>$ to denote the path. The degree of vertex v in G, denoted by $d_G(v) = |N(v)|$. The minimum degree of G is denoted by $\delta(G) = \min \{d_G(v) | v \in V(G)\}$. We use $<n>$ to denote the set $\{1, 2, \ldots, n\}$. An independent set in a graph is a set of pairwise nonadjacent vertices. A graph $G = (V_0 \cup V_1, E)$ is bipartite if $V_0 \cap V_1 = \varnothing$ and $E \subseteq \{(u, v) | u \in V_0 \text{ and } v \in V_1\}$. Star graphs are bipartite graphs. In [18], Wong introduced the concept of Hamiltonian laceability for the class of bipartite graphs. A bipartite graph $G = (V_0 \cup V_1, E)$ with $|V_0| = |V_1|$ is *Hamiltonian laceable* if there is a Hamiltonian path between every vertex of V_0 and every vertex of V_1, where V_0 and V_1 are the two partite sets of G. We note that any path between two vertices of the same partite set has length at most $|V_0| + |V_1| - 2$.

It is meaningful to extend the concept of Hamiltonian laceability so that a longest path between every two vertices of the same partite set is required. As [7], we say that a Hamiltonian laceable graph $G = (V_0 \cup V_1, E)$ is *strongly Hamiltonian laceable* if G, additionally, has the property that there is a path of length $|V_0| + |V_1| - 2$ between every two vertices of the same partite set. In other words, there is a longest path between every two vertices of a strongly Hamiltonian laceable graph. Lewinter and Widulski [12] introduced extended concept of *hyper Hamiltonian laceability*. G is *hyper Hamiltonian laceable* if it is Hamiltonian laceable and, for any vertex $v \in V_i$, there is a Hamiltonian path in $G - v$ between any two vertices in V_{1-i}, $i \in \{0, 1\}$.

Throughout this paper, the paired terms network and graph, node and vertex, and link and edge are used interchangeably. The following is a formal definition of star graph.

Definition 1. A n-dimensional star graph is denoted by S_n, which is a graph with $V(S_n) = \{a_1a_2 \ldots a_n \mid a_1a_2 \ldots a_n$ is a permutation of $1, 2, \ldots, n\}$, and $E(S_n) = \{(a_1a_2 \ldots a_n, a_ia_2 \ldots a_{i-1}a_1a_{i+1} \ldots a_n) \mid a_1a_2 \ldots a_n \in V(S_n)$ and $2 \leq i \leq n\}$.

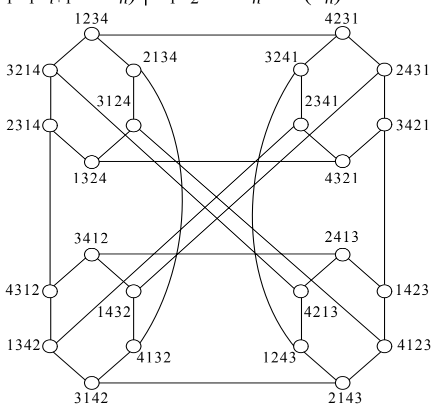

Fig.1. The structure of S_4

S_n has $n!$ nodes in which each node is one permutation of $1, 2, \ldots, n$. Two nodes are adjacent if and only if they can be obtained from each other by swapping the leftmost number with one of the other $n - 1$ numbers. Apparently, S_n is regular of degree $n - 1$. The structures of S_1, S_2, and S_3 are a node, an edge, and a cycle with a length of six, respectively. The structure of S_4 is shown in Fig. 1. For convenience, we say that the edge between nodes $a_1a_2 \ldots a_n$ and $a_ia_2 \ldots a_{i-1}a_1a_{i+1} \ldots a_n$ is along the dimension i, where $2 \leq i \leq n$. If $v \in V(S_n)$, we use $e(v)$ to denote the set of incident edges of v in S_n and $e_i(v)$ denote the incident edge of v along dimension i. $E_i(S_n)$ denote the set of edges along dimension i in S_n. An S_n is a recursive structure that contains many smaller stars, which are referred to as embedded S_rs of S_n, where $1 \leq r \leq n$. We use $S_k^{b_{k+1} \ldots b_n}$ to denote the induced subgraph of S_n with vertex set $\{a_1a_2 \ldots a_n \mid a_{k+1} \ldots a_n = b_{k+1} \ldots b_n\}$. Note that S_n consists of n disjoint embedded S_{n-1}s connected with the edges along dimension i. In this situation, we say that we have partitioned S_n over dimension i. There are $(n - 2)!$ edges between S_{n-1}^i and S_{n-1}^j for $1 \leq i \neq j \leq n$. Conveniently, we use $E^{i,j}(S_n)$ to denote the set of these edges.

The following lemmas, which are results in [13], will also be used very often.

Lemma 1. [13] S_n is $(n - 3)$-edge fault tolerant strongly Hamiltonian laceable, where $n \geq 4$. This means that in S_n with at most $n - 3$ faulty edges, we can find a Hamiltonian path between every two distinct nodes of different partite sets and a path of length $n! - 2$ between every two distinct nodes of the same partite set.

Lemma 2. [13] The star graph S_n is hyper Hamiltonian laceable for $n \geq 4$.

Meanwhile, nodes and edges that are not faulty are referred to as healthy nodes/edges.

3: MAIN RESULT

In this section, we would show that for any S_n $(n \geq 4)$ with $\leq 2n - 7$ edge faults in which each node is incident to at least two healthy edges, a fault-free path of length $n! - 1$(or $n! - 2$) between two arbitrary nodes in different (or the same) partite sets could be obtained.

The basic idea uses an inductive proof. First, S_{n+1} will be partitioned into $n + 1$ disjoint S_ns such that every node is incident with at least two healthy edges in each S_n. Then, we can combine the paths of each S_n into a new path in the S_{n+1}. Hence, we should have a method to partition star graph. The following lemma in [6] will be used in our proof.

Lemma 3. [6] Let $F \subset E(S_{n+1})$, $|F| \leq 2n - 5$ and $\delta(S_{n+1} - F) = 2$, where $n \geq 3$. We can partition S_{n+1} over some dimension $i \in <n+1> - \{1\}$ such that for all $q \in <n+1>$, $\delta(<*^{i-1}q*^{n+1-i}>_n - F) = 2$ and $|E_i(S_{n+1}) \cap F| \geq 1$.

We also need the following lemmas to prove our main theorem.

Lemma 4. Let $F \subset E(S_{n+1})$, $|F| \leq 2n - 5$, $q \in <n+1>$ and $\delta(S_{n+1} - F) = 2$, where $n \geq 3$. Suppose that there is an s-t path P with length $n! - 1$ or $n! - 2$ in $S_n^q - F$. Then there exists an edge (v, u) of P such that $e_{n+1}(v)$, $e_{n+1}(u) \in E_{n+1}(S_{n+1}) - F$.

Proof. Since the length of P is $n! - 1$or $n! - 2$, we have at least $n! - 2$ choices. If none of the edges of P meet the requirements of such an edge (v, u), then $|E_{n+1}(S_{n+1}) \cap F| \geq \lceil (n! - 2)/2 \rceil$. (Because an edge in $E_{n+1}(S_{n+1})$ eliminates two edges of P.) Moreover, $\lceil (n! - 2)/2 \rceil > 2n - 5$ for $n \geq 3$. This contracts $|F| \leq 2n - 5$. Therefore, we can always find such (v, u) in P. □

Lemma 5. Given a fixed n, let $F \subseteq E(S_n)$, $A \subseteq <n>$, and let S_A denote the subgraph of S_n induced by $V(\bigcup_{k \in A} S_{n-1}^k)$. For all distinct integers $i, j \in A$, if $|E^{i,j}(S_n) \cap F| < (n - 2)!/2$, and $S_{n-1}^{i}-F$, $S_{n-1}^{j}-F$ are strongly Hamiltonian laceable, then S_A-F is strongly Hamiltonian laceable.

Proof. For all distinct integers $i, j \in A$ and $|E^{i,j}(S_n) \cap F| < (n-2)!/2$, there are at least two edges (u^i, v^j), $(u^{i'}, v^{j'}) \in E^{i,j}(S_n) - F$ such that $u^i, u^{i'} \in V(S_{n-1}^{i})$, $v^j, v^{j'} \in V(S_{n-1}^{j})$, and u^i and $u^{i'}$(v^j and $v^{j'}$, respectively) are in different partite sets. Suppose $A = \{k_1, k_2, \ldots, k_m\}$, where $m \le n$ and each k is distinct. Let s and t be arbitrary two nodes in S_A-F. We want to construct a longest s-t path in S_A–F. Two cases are considered:

Case 1. $s \in V(S_{n-1}^{a})$ and $t \in V(S_{n-1}^{b})$, where $a, b \in A$ and $a \neq b$. Without loss of generality, let $k_1 = a$ and $k_m = b$. Let $u^1 = s$. From above discussion we can find edges (v^1, u^2), (v^2, u^3), (v^3, u^4), … , $(v^{m-1}, u^m) \in E_n(S_n) - F$, where $u^i, v^i \in V(S_{n-1}^{k_i})$, u^i and v^i are in different partite sets, for all $i \in$ <m>. Since $S_{n-1}^{k_i} - F$ is strongly Hamiltonian laceable for all $k_i \in A$, there is a u^i-v^i Hamiltonian path P_i in $S_{n-1}^{k_i} - F$ for all $i \in$ <$m-1$>, and a u^m-t path P_m with a length $(n-1)!-1$ ($(n-1)!-2$, respectively) in $S_{n-1}^{k_m} - F$ if s and t are in different partite set (in the same partite set, respectively). An s-t path Q in S_A-F is constructed as follows (see Fig. 2):

< $s, P_1, v^1, u^2, P_2, v^2, \ldots, u^m, P_m, t$ >.

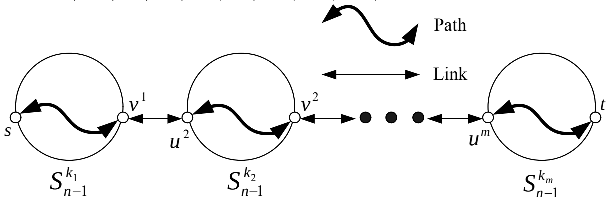

Fig.2. *s* and *t* are in different substars of Lemma 5.

The length of Q is $(m-1) \times ((n-1)!-1) + (m-1) + ((n-1)!-1) = m \times (n-1)! - 1$ if s and t are in different partite set, and the length of Q is $(m-1) \times ((n-1)!-1) + (m-1) + ((n-1)!-2) = m \times (n-1)! - 2$ if s and t are in the same partite set. So S_A-F is strongly Hamiltonian laceable.

Case 2. $s, t \in V(S_{n-1}^{q})$ for some $q \in A$. Without loss of generality, let $k_1 = q$. Since $S_{n-1}^{k_1} - F$ is strongly Hamiltonian laceable, there is an s-t path P_1 with a length $(n-1)!-1$ ($(n-1)!-2$, respectively) in $S_{n-1}^{k_1} - F$ if s and t are in different partite set (in the same partite set, respectively). In addition, by Lemma 4, there exists an edge (v^1, u^1) of P_1 such that (v^1, u^2), $(v^m, u^1) \in E_n(S_n) - F$, where $u^2 \in V(S_{n-1}^{c})$ and $v^m \in V(S_{n-1}^{d})$, where $c, d \in A - \{k_1\}$. Without loss of generality, let $k_2 = c$ and $k_m = d$. Similar to Case 1, we can find (v^2, u^3), (v^3, u^4), … , $(v^{m-1}, u^m) \in E_n(S_n) - F$, where $u^i, v^i \in V(S_{n-1}^{k_i})$, u^i and v^i are in different partite sets, for all $i \in$ <m> $- \{1\}$. Since $S_{n-1}^{k_i} - F$ is strongly Hamiltonian laceable for all $k_i \in A$, there is a u^i-v^i Hamiltonian path P_i in $S_{n-1}^{k_i} - F$ for all $i \in$ <m> $- \{1\}$. Additionally, let P and P' denote s-v^1 and u^1-t subpaths of P_1. An s-t path Q in S_A-F is constructed as follows (see Fig. 3):

< $s, P, v^1, u^2, P_2, v^2, \ldots, u^m, P_m, v^m, u^1, P', t$ >.

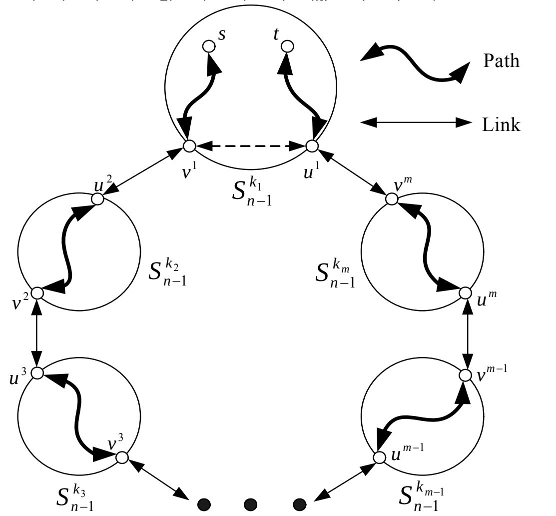

Fig.3. *s* and *t* are in the same substar of Lemma 5.

The length of Q is $(m-1) \times ((n-1)!-1) + m + ((n-1)!-1) - 1 = m \times (n-1)! - 1$ if s and t are in different partite set, and the length of Q is $(m-1) \times ((n-1)!-1) + m + ((n-1)!-2) - 1 = m \times (n-1)! - 2$ if s and t are in the same partite set. So S_A-F is strongly Hamiltonian laceable. □

Lemma 6. Let $s, t \in V(S_n)$. For any edge $e \neq (s, t)$ in $S_n(n \ge 4)$, there exists a s-t path including e with a length of $n!-1$ (or $n!-2$) in S_n if s and t are in different partite sets (or in the same partite set).

Proof. We can use a computer program to check that the result is true for S_4, and then proceed by induction on n. As per our induction hypothesis, assume that the result holds for S_n for some $n \ge 4$. Now consider S_{n+1}. Let $e = (a, b)$. Since a and b are adjacent, they differ only in the 1st position and the ith position for some $i \in$ <n+1> $- \{1\}$. We can partition S_{n+1} over some dimension $j \in$ <n+1> $- \{1, i\}$, hence make a and b in the same S_n. Without loss of generality, let $j = n+1$, that is, each embedded S_n is S_n^r for all $r \in$ <n+1>. Suppose $e \in E(S_n^k)$ for some $k \in$ <n+1>. Five cases are considered:

Case 1. $s \in V(S_n^k)$, $t \in V(S_n^q)$ for some $q \in$ <n+1> $- \{k\}$. Let $k = i_1$, $q = i_{n+1}$. Let $(v^1, u^2) \in E_{n+1}(S_{n+1})$ such that $v^1 \in V(S_n^{i_1}) - \{a, b\}$, $u^2 \in V(S_n^{i_2})$, v^1 and s are in different partite sets, where $i_2 \in$ <n+1> $- \{i_1, i_{n+1}\}$. With the induction hypothesis, there is a s-v^1 Hamiltonian path P_1 including e for $S_n^{i_1}$. Let $A =$ <n+1> $- \{i_1\}$. Since each S_n is strongly Hamiltonian laceable, by Lemma 5, S_A is strongly Hamiltonian laceable. Hence there is an u^2-t path P with a length of $n \times n! - 1$ (or $n \times n! - 2$) in S_A if s and t are in different partite sets (or in the same partite set). An s-t path including e in S_{n+1} is constructed as follows (see Fig. 4):

< s, P_1, v^1, u^2, P, t >.

The length of the s-t path is $(n \times n! - 1) + 1 + (n! - 1) = (n+1)! - 1$ if s and t are in different partite sets, and the length of the s-t path is $(n \times n! - 2) + 1 + (n! - 1) = (n + 1)! - 2$ if s and t are in the same partite set.

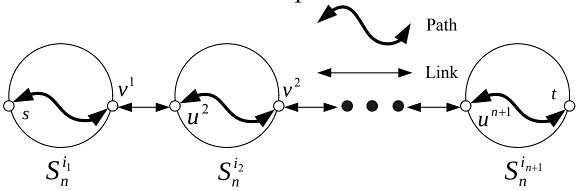

Fig.4. *s* and *t* are in different substars.

Case 2. $t \in V(S_n^k)$, $s \in V(S_n^q)$ for some $q \in <n+1> - \{k\}$. The construction of an s-t path including e in S_{n+1} is similar to that of Case 1.

Case 3. $s \in V(S_n^p)$, $t \in V(S_n^q)$, where $p, q \in <n+1> - \{k\}$ and $p \neq q$. Let $i_1 = p$, $i_{n+1} = q$, $i_2 = k$. Let (v^1, u^2), $(v^2, u^3) \in E_{n+1}(S_{n+1})$ such that $v^1 \in V(S_n^{i_1})$, $u^2, v^2 \in V(S_n^{i_2}) - \{a, b\}$, $u^3 \in V(S_n^{i_3})$, v^1 and s, v^2 and u^2 are in different partite sets, where $i_3 \in <n+1> - \{i_1, i_2, i_{n+1}\}$. By Lemma 1, there is a s-v^1 Hamiltonian path P_1 for $S_n^{i_1}$. With the induction hypothesis, there is a u^2-v^2 Hamiltonian path P_2 including e for $S_n^{i_2}$. Let $A = <n+1> - \{i_1, i_2\}$. Since each S_n is strongly Hamiltonian laceable, by Lemma 5, S_A is strongly Hamiltonian laceable. Hence there is an u^3-t path P with a length of $(n - 1) \times n! - 1$ (or $(n - 1) \times n! - 2$) in S_A if s and t are in different partite sets (or in the same partite set). An s-t path including e in S_{n+1} is constructed as follows (see Fig. 4):

$< s, P_1, v^1, u^2, P_2, v^2, u^3, P, t >$.

The length of the s-t path is $((n - 1) \times n! - 1) + 2 + 2 \times (n! - 1) = (n + 1)! - 1$ if s and t are in different partite sets, and the length of the s-t path is $((n - 1) \times n! - 2) + 2 + 2 \times (n! - 1) = (n + 1)! - 2$ if s and t are in the same partite set.

Case 4. $s, t \in V(S_n^k)$. Let $k = i_1$. With the induction hypothesis, there is a s-t path P_1 including e in $S_n^{i_1}$ with a length $n! - 1$ (or $n! - 2$) when s and t are in different partite sets (or in the same partite set). Moreover, with Lemma 4, we can find an edge $(v^1, u^1) \neq e$ of P_1 such that (v^1, u^2), $(v^{n+1}, u^1) \in E_{n+1}(S_{n+1})$, where $u^2 \in V(S_n^{i_2})$, $v^{n+1} \in V(S_n^{i_{n+1}})$, and $i_2, i_{n+1} \in <n+1> - \{i_1\}$. Let $A = <n+1> - \{i_1\}$. Since u^2 and v^{n+1} are in different partite sets, by Lemma 5, there is an u^2-v^{n+1} Hamiltonian path P_A in S_A. Additionally, let P and P' denote s-v^1 and u^1-t subpaths of P_1. An s-t path including e in S_{n+1} is constructed as follows (see Fig. 5):

$< s, P, v^1, u^2, P_A, v^{n+1}, u^1, P', t >$.

The length of the s-t path is $(n! - 1) - 1 + n \times (n! - 1) + (n + 1) = (n + 1)! - 1$ if s and t are in different partite sets, and the length of the s-t path is $(n! - 2) - 1 + n \times (n! - 1) + (n + 1) = (n + 1)! - 2$ if s and t are in the same partite set.

Case 5. $s, t \in V(S_n^q)$ for some $q \in <n+1> - \{k\}$. Let $i_1 = q$, and $i_2 = k$. By Lemma 1, there is an s-t path P_1 in $S_n^{i_1}$ with a length $n! - 1$ (or $n! - 2$) when s and t are in different partite sets (or in the same partite set). Moreover, with Lemma 4, we can find an edge (v^1, u^1) of P_1 such that (v^1, u^2), $(v^{n+1}, u^1) \in E_{n+1}(S_{n+1})$, where $u^2 \in V(S_n^{i_2}) - \{a, b\}$, $v^{n+1} \in V(S_n^{i_{n+1}})$, and $i_{n+1} \in <n+1> - \{i_1, i_2\}$. Let $(v^2, u^3) \in E_{n+1}(S_{n+1})$ such that $v^2 \in V(S_n^{i_2}) - \{a, b\}$, v^2 and u^2 are in different partite sets, $u^3 \in V(S_n^{i_3})$, where $i_3 \in <n+1> - \{i_1, i_2, i_{n+1}\}$. With the induction hypothesis, there is a u^2-v^2 Hamiltonian path P_2 including e for $S_n^{i_2}$. Let $A = <n+1> - \{i_1, i_2\}$. Since u^3 and v^{n+1} are in different partite sets, by Lemma 5, there is an u^3-v^{n+1} Hamiltonian path P_A in S_A. Additionally, let P and P' denote s-v^1 and u^1-t subpaths of P_1. An s-t path including e in S_{n+1} is constructed as follows (see Fig. 5):

$< s, P, v^1, u^2, P_2, v^2, u^3, P_A, v^{n+1}, u^1, P', t >$.

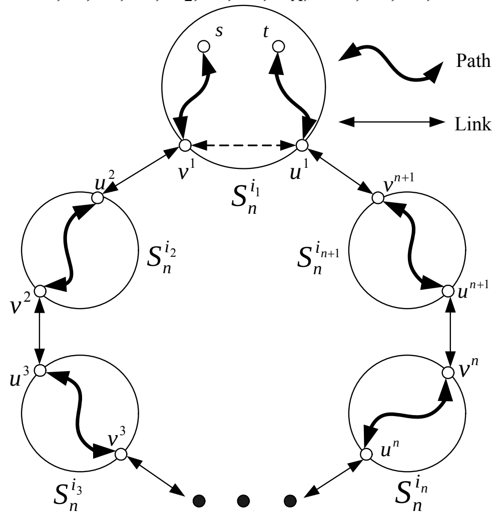

Fig.5. *s* and *t* are in the same substar.

The length of the s-t path is $(n! - 1) - 1 + n \times (n! - 1) + (n + 1) = (n + 1)! - 1$ if s and t are in different partite sets, and the length of the s-t path is $(n! - 2) - 1 + n \times (n! - 1) + (n + 1) = (n + 1)! - 2$ if s and t are in the same partite set □

Lemma 7. Let (s, t), (u, v) denote two distinct edges in $S_n (n \geq 3)$. Then there exists a Hamiltonian cycle in S_n including (s, t) and (u, v).

Proof. The result is trivial for $n = 3$. When $n \geq 4$, since s and t are in different partite sets, by Lemma 6, there exists a Hamiltonian path P from s to t including (u, v). Then P and (s, t) form a Hamiltonian cycle in S_n. □

Lemma 8. Let $u \in V(S_{n-1}^q)$ for some $q \in <n>$. Then $|e_n(N(u)) \cap E^{i,q}(S_n)| = 1$ for all $i \in <n> - \{q\}$.

Proof. Suppose $u = a_1a_2 \ldots a_n$, where $\{a_1, a_2, \ldots, a_n\} = <n>$ and $a_n = q$. Let $v \in N(u)$. This means $v = a_ja_2 \ldots a_{j-1}a_1a_{j+1} \ldots a_n$ for some $2 \leq j \leq n$, hence $e_n(v)$ incident to $S_{n-1}^{a_j}$. For $w \neq v \in N(u)$, $w = a_ka_2 \ldots a_{k-1}a_1a_{k+1} \ldots a_n$ for some $2 \leq k \leq n$, then $k \neq j$. Since $|N(u)| = n - 1$ and $|N(u) \cap V(S_{n-1}^q)| = n - 2$, if $v \in N(u) - V(S_{n-1}^q)$, then $e_n(v) = e_n(u) = (u, v) \in E^{a_1,q}(S_n)$. Else $e_n(v) \in E^{a_j,q}(S_n)$, where $2 \leq j \leq n - 1$. So we have $|e_n(N(u)) \cap E^{i,q}(S_n)| = 1$ for all $i \in <n> - \{q\}$. □

In the following, we start our main proof.

Theorem 1. Let $F \subset E(S_n)$, $n \geq 4$, where $|F| \leq 2n - 7$ and $\delta(S_n - F) = 2$. Then, $S_n - F$ is strongly Hamiltonian laceable.

Proof. We proceed by induction on n. Since $n - 3 = 2n - 7$ when $n = 4$, according to Lemma 1, the theorem holds for $n = 4$. As per our induction hypothesis, assume that the result holds for S_n for some $n \geq 4$. Consider S_{n+1} with $|F| \leq 2n - 5$ and $\delta(S_{n+1} - F) = 2$. By Lemma 3, we can partition S_{n+1} over some dimension $i \in <n+1> - \{1\}$ such that for all $q \in <n+1>$, $\delta(<*^{i-1}q*^{n+1-i}>_n - F) = 2$ and $|E_i(S_{n+1}) \cap F| \geq 1$. Without loss of generality, let $i = n + 1$, that is, each embedded S_n is S_n^q for all $q \in <n+1>$. Since $|E_{n+1}(S_{n+1}) \cap F| \geq 1$, we have $|E(S_n^q) \cap F| \leq 2n - 6$, for all $q \in <n+1>$. Let s and t be arbitrary two nodes in $S_{n+1}-F$. We want to construct a longest s-t path in $S_{n+1}-F$. Two cases are considered:

Case 1. $|F \cap E(S_n^q)| \leq 2n - 7$, for all $q \in <n+1>$. We have $\sum_{i \neq j} |E^{i,j}(S_{n+1}) \cap F| \leq 2n - 5 \leq \frac{((n+1)-2)!}{2}$ for all i, $j \in <n+1>$ and $i \neq j$, where $n \geq 4$ since $|F| \leq 2n - 5$. If there are two distinct integers i', $j' \in <n+1>$ such that $|E^{i',j'}(S_{n+1}) \cap F| = ((n+1)-2)!\,/2$, then we have $n = 4$ and $|E^{i,j}(S_{n+1}) \cap F| = 0 < ((n+1)-2)!\,/2$ for all $i, j \in <n+1>$, where $i \neq j$ and $\{i', j'\} \neq \{i, j\}$. Two cases are further considered:

Case 1.1. $n > 4$. Let $A = <n+1>$. In this case, we have $|E^{i,j}(S_{n+1}) \cap F| < ((n+1)-2)!\,/2$ for all distinct $i, j \in A$. With the induction hypothesis, $S_n^q - F$ is strongly Hamiltonian laceable for all $q \in A$. By Lemma 5, $S_{n+1} - F$ is strongly Hamiltonian laceable.

Case 1.2. $n = 4$. If $|E^{i,j}(S_5) \cap F| < (n-1)!\,/\,2$ for all distinct integers i, $j \in <5>$, then the discussion is the same as when $n > 4$. We assume that $|E^{i',j'}(S_5) \cap F| = ((4+1)-2)!\,/2 = 3$, for some distinct integers $i', j' \in <5>$. First, consider that $s \in V(S_4^a)$ and $t \in V(S_4^b)$, where a, $b \in <5>$ and $a \neq b$. Let $i_1 = a$ and $i_5 = b$. We can always find $r \in <4>$ such that $i' = i_r$ and $j' \neq i_{r+1}$. Then the construction of an s-t path in S_5 is similar to Case 1 of Lemma 5. Now, consider that $s, t \in V(S_4^q)$ for some $q \in <5>$. Let $i_1 = q$. If $i_1 \in \{i', j'\}$, we can let $i_3 \in \{i', j'\}$ and $i_3 \neq i_1$. Then $|E^{i_1,i_3}(S_{n+1}) \cap F| = 3$ and $|E^{i,j}(S_5) \cap F| = 0$ for all $i, j \in <5>$, where $i \neq j$ and $\{i, j\} \neq \{i_1, i_3\}$. So the rest of the construction is similar to Case 2 of Lemma 5. If $i_1 \notin \{i', j'\}$, then no edge in $E_5(S_5) \cap F$ connected to $S_4^{i_1}$. Let (v^1, u^1) be an edge of the s-t path of length $4! - 1$ or $4! - 2$ in $S_4^{i_1}$, where (v^1, u^2), $(v^5, u^1) \in E_5(S_5)$, $u^2 \in V(S_4^{i'})$, $v^5 \in V(S_4^{j'})$. Let $i_2 = i'$ and $i_5 = j'$. Then the rest of the construction is similar to Case 2 of Lemma 5.

Case 2. $|F \cap E(S_n^k)| = 2n - 6$ for some $k \in <n+1>$. Because $|F| \leq 2n - 5$ and $|E_{n+1}(S_{n+1}) \cap F| = 1$, we have $|E^{i,j}(S_{n+1}) \cap F| \leq 1 < ((n+1)-2)!\,/2$ for all distinct $i, j \in <n+1>$, and $|F \cap E(S_n^q)| = 0$ for all $q \in <n+1> - \{k\}$, where $n \geq 4$. Five cases are further considered:

Case 2.1. $s, t \in V(S_n^k)$. In this case, let $i_1 = k$ and $(v^1, u^1) \in E(S_n^{i_1}) \cap F$ such that $e_{n+1}(v^1)$, $e_{n+1}(u^1) \notin F$, where $s \neq v^1$ or u^1. Since $|S_n^{i_1} \cap (F - \{(v^1, u^1)\})| = 2n - 7$, with the induction hypothesis, there is an s-t path P in $S_n^{i_1} - (F - \{(v^1, u^1)\})$ with length $n! - 1$ ($n! - 2$, repectively) if s and t are in different partite sets (in the same partite set, respectively). Assume that P contains (v^1, u^1) (otherwise, the discussion is easier). The construction of an s-t path of S_{n+1} is similar to Case 2 of Lemma 5.

Case 2.2. $s \in V(S_n^k)$ and $t \in V(S_n^q)$ for some $q \in <n+1> - \{k\}$. Let $(v^+, u^+) \in E(S_n^k) \cap F$ such that $e_{n+1}(v^+)$, $e_{n+1}(u^+) \notin F$, where $s \neq v^+$ or u^+. Let $i_1 = k$ and $i_{n+1} = q$. Two cases are further considered:

Case 2.2.1. Neither v^+ nor u^+ connects to $S_n^{i_{n+1}}$. Let $v^1 \in V(S_n^{i_1}) - \{s\}$ such that v^1 and t are in the same partite set, $e_{n+1}(v^1) \notin F$. And let (v^+, u^2), (v^3, u^+), $(v^1, u^4) \in E_{n+1}(S_{n+1}) - F$, where $u^2 \in V(S_n^{i_2})$, $v^3 \in V(S_n^{i_3})$, $u^4 \in V(S_n^{i_4})$, and $i_2, i_3, i_4 \in <n+1> - \{i_1, i_{n+1}\}$. Note that $|S_n^k \cap (F - \{(v^+, u^+)\})| = 2n - 7$. With the induction hypothesis, there is an s-v^1 path P with a length of $n! - 1$ (or $n! - 2$) in $S_n^{i_1}$ if s and t are in different partite sets (or in the same partite set). Assume that P contains (v^+, u^+) (otherwise, the discussion is easier). Let $A = \{i_2, i_3\}$ and $B = \{i_4, i_5, \ldots, i_{n+1}\}$, where $\{i_5, i_6, \ldots, i_n\} = <n+1> - \{i_1, i_2, i_3, i_4, i_{n+1}\}$. Since u^2 and v^3, u^4 and t are in different partite sets, by Lemma 5, there is an u^2-v^3 Hamiltonian path P_1 in S_A-F and an u^4-t Hamiltonian path P_2 in S_B-F. An s-t path in S_{n+1} is constructed as follows (see Fig. 6):

$< s, \ldots, v^+, u^2, P_1, v^3, u^+, \ldots, v^1, u^4, P_2, t >$.

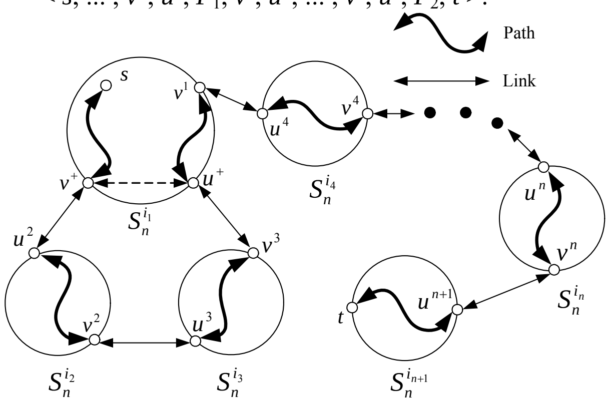

Fig.6. Illustration for case 2.2.1.

The length of the s-t path is $(n! - 1) - 1 + (n! - 1) \times n + (n + 1) = (n + 1)! - 1$ if s and t are in different partite sets, and the length of the s-t path is $(n! - 2) - 1 + (n! - 1) \times n + (n + 1) = (n + 1)! - 2$ if s and t are in the same partite set.

Case 2.2.2. One of v^+ or u^+ connects to $S_n^{i_{n+1}}$. Without loss of generality, we may assume u^+ connects to $S_n^{i_{n+1}}$. Two cases are further considered:

Case 2.2.2.1. u^+ does not connect to t. Let (u^+, v^-), $(v^+, u^2) \in E_{n+1}(S_{n+1}) - F$, where $u^2 \in V(S_n^{i_2})$, $v^- \in V(S_n^{i_{n+1}})$ and $i_2 \in \langle n+1\rangle - \{i_1, i_{n+1}\}$. We can find $(u^1, v^n) \in E_{n+1}(S_{n+1}) - F$, where $u^1 \neq s \in V(S_n^{i_1})$, u^1 and t are in the same partite set, $v^n \in V(S_n^{i_n})$, and $i_n \in \langle n+1\rangle - \{i_1, i_2, i_{n+1}\}$. Similar to case 2.2.1, assume there is an s-u^1 path in $S_n^{i_1}$ containing (v^+, u^+) with a length of $n! - 1$ (or $n! - 2$) if s and t are in different partite sets (or in the same partite set). By Lemma 8, we can find $(v^-, w) \in E(S_n^{i_{n+1}})$ with $(w, v^2) \in E_{n+1}(S_{n+1})$, where $v^2 \in V(S_n^{i_2})$. First, consider the case $w \neq t$ and $(w, v^2) \notin F$. Let $u^{n+1} = w$ and $(t, v^{n+1}) \in E(S_n^{i_{n+1}})$ such that $(v^{n+1}, u^3) \in E_{n+1}(S_{n+1}) - F$, where $u^3 \in V(S_n^{i_3})$, $i_3 \in \langle n+1\rangle - \{i_1, i_2, i_n, i_{n+1}\}$ (since $|F \cap E(S_n^{i_{n+1}})| = 0$ and $|E_{n+1}(S_{n+1}) \cap F| = 1$, with Lemma 8, we can find such v^{n+1}.) Additionally, with Lemma 7, there exists a Hamiltonian cycle including (t, v^{n+1}) and (v^-, u^{n+1}). Let $A = \{i_3, i_4, \ldots, i_n\}$, where $\{i_4, i_5, \ldots, i_{n-1}\} = \langle n+1\rangle - \{i_1, i_2, i_3, i_n, i_{n+1}\}$. Since u^2 and v^2, u^3 and v^n are in different partite sets, by Lemma 1 and Lemma 5, there is an u^2-v^2 Hamiltonian path P_1 in $S_n^{i_2}$ and an u^3-v^n Hamiltonian path P_2 in $S_A - F$. An s-t path in S_{n+1} is constructed as follows (see Fig. 7a):

$< s, \ldots, v^+, u^2, P_1, v^2, u^{n+1}, \ldots, v^{n+1}, u^3, P_2, v^n, u^1, \ldots, u^+, v^-, \ldots, t >$.

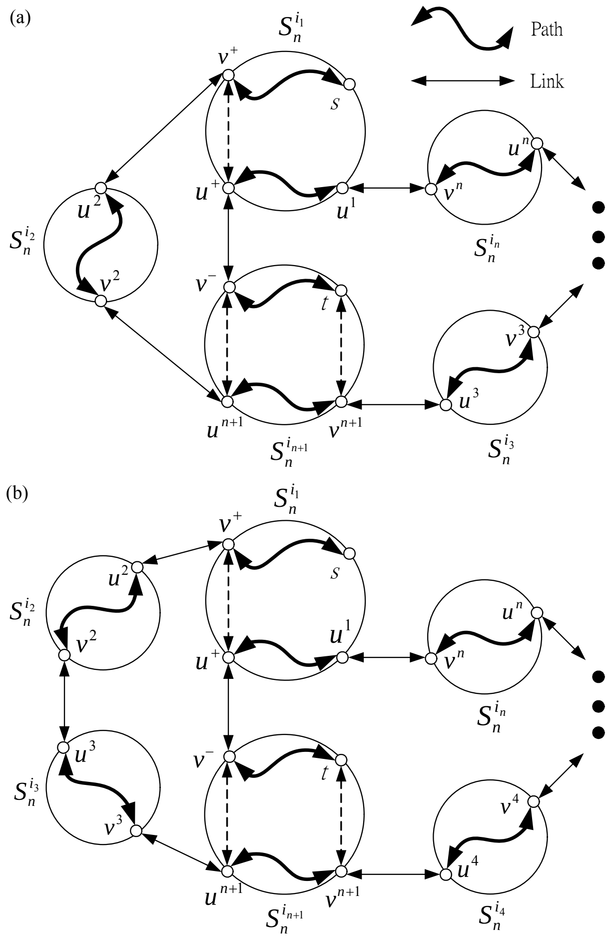

Fig.7. Illustration for case 2.2.2.1.

Now, consider that $w = t$ or $(w, v^2) \in F$. Since $|F \cap E(S_n^{i_{n+1}})| = 0$ and $|E_{n+1}(S_{n+1}) \cap F| = 1$, by Lemma 8, we can always find $(v^-, w') \in E(S_n^{i_{n+1}})$ such that $w' \neq t$ and $(w', v^3) \in E_{n+1}(S_{n+1}) - F$, where $v^3 \in V(S_n^{i_3})$, $i_3 \in \langle n+1\rangle - \{i_1, i_2, i_n, i_{n+1}\}$. Let $u^{n+1} = w'$ and $(t, v^{n+1}) \in E(S_n^{i_{n+1}})$ such that $(v^{n+1}, u^4) \in E_{n+1}(S_{n+1}) - F$, where $u^4 \in V(S_n^{i_4})$, $i_4 \in \langle n+1\rangle - \{i_1, i_2, i_3, i_n, i_{n+1}\}$ when $n > 4$ (the discussion is similar when $n = 4$). By Lemma 7, there exists a Hamiltonian cycle including (t, v^{n+1}) and (v^-, u^{n+1}). Let $A = \{i_2, i_3\}$ and $B = \{i_4, i_5, \ldots, i_n\}$, where $\{i_5, i_6, \ldots, i_{n-1}\} = \langle n+1\rangle - \{i_1, i_2, i_3, i_4, i_n, i_{n+1}\}$. Since u^2 and v^3, u^4 and v^n are in different partite sets, by Lemma 5, there is an u^2-v^3 Hamiltonian path P_1 in $S_A - F$ and an u^4-v^n Hamiltonian path P_2 in $S_B - F$. An s-t path in S_{n+1} is constructed as follows (see Fig. 7b):

$< s, \ldots, v^+, u^2, P_1, v^3, u^{n+1}, \ldots, v^{n+1}, u^4, P_2, v^n, u^1, \ldots, u^+, v^-, \ldots, t >$.

The length of the s-t path is $2 \times ((n! - 1) - 1) + (n! - 1) \times (n - 2) + 5 + (n - 3) + (n! - 1) = (n + 1)! - 1$ if s and t are in different partite set, and the length of the s-t path is $2 \times ((n! - 1) - 1) + (n! - 1) \times (n - 2) + 5 + (n - 3) + (n! - 2) = (n + 1)! - 2$ if s and t are in the same partite set.

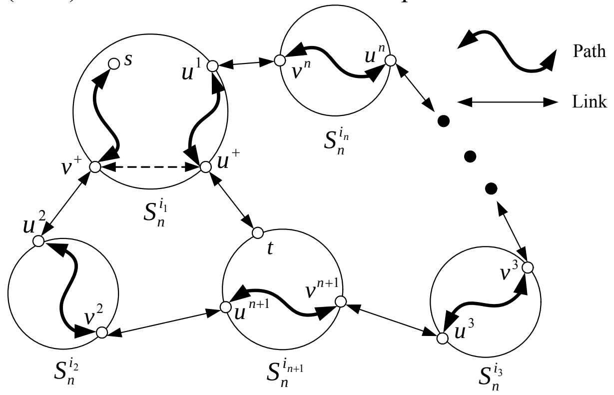

Fig.8. Illustration for case 2.2.2.2.

Case 2.2.2.2. u^+ connects to t. Let $(v^+, u^2) \in E_{n+1}(S_{n+1}) - F$, where $u^2 \in V(S_n^{i_2})$ and $i_2 \in \langle n+1\rangle - \{i_1, i_{n+1}\}$. We can find $(u^1, v^n) \in E_{n+1}(S_{n+1}) - F$ with $u^1 \neq s \in V(S_n^{i_1})$, $v^n \in V(S_n^{i_n})$, u^1 and t are in the same partite set, where $i_n \in \langle n+1\rangle - \{i_1, i_2, i_{n+1}\}$. Note that $|S_n^k \cap (F - \{(v^+, u^+)\})| = 2n - 7$, with the induction hypothesis, there is an s-u^1 path P in $S_n^k - (F - \{(v^+, u^+)\})$ with a length $n! - 1$ (or $n! - 2$) when s and t are in different partite sets (or in the same partite set). Assume that P contains (v^+, u^+) (otherwise, the discussion is easier). Let (v^2, u^{n+1}), $(v^{n+1}, u^3) \in E_{n+1}(S_{n+1}) - F$, where $v^2 \in V(S_n^{i_2})$, $u^3 \in V(S_n^{i_3})$, u^{n+1}, $v^{n+1} \in V(S_n^{i_{n+1}})$, v^2 and u^2 are in different partite sets, u^{n+1} and v^{n+1} are in the same partite set, and $i_3 \in \langle n+1\rangle - \{i_1, i_2, i_n, i_{n+1}\}$. By Lemma 1 and Lemma 2, there is an u^2-v^2 Hamiltonian path P_1 for $S_n^{i_2}$ and an u^{n+1}-v^{n+1} Hamiltonian path P_2 for $S_n^{i_{n+1}} - \{t\}$. Let $A = \{i_3, i_4, \ldots, i_n\}$, where $\{i_4, i_5, \ldots, i_{n-1}\} = \langle n+1\rangle - \{i_1, i_2, i_3, i_n, i_{n+1}\}$. Since u^3 and v^n

are in different partite sets, by Lemma 5, there is an u^3-v^n Hamiltonian path P_3 in S_A–F. An s-t path in S_{n+1} is constructed as follows (see Fig. 8):

$< s, \ldots, v^+, u^2, P_1, v^2, u^{n+1}, P_2, v^{n+1}, u^3, P_3, v^n, u^1, \ldots, u^+, t >$.

The length of the s-t path is $(n! - 1) \times (n - 1) + (n! - 2) + (n + 2) + ((n! - 1) - 1) = (n + 1)! - 1$ if s and t are in different partite set, and the length of the s-t path is $(n! - 1) \times (n - 1) + (n! - 2) + (n + 2) + ((n! - 2) - 1) = (n + 1)! - 2$ if s and t are in the same partite set.

Case 2.3. $t \in V(S_n^k)$ and $s \in V(S_n^q)$ for some $q \in$ <n+1> – $\{k\}$. Let $i_1 = q$ and $i_{n+1} = k$. The construction of an s-t path in S_{n+1} is similar to that of Case 2.2.

Case 2.4. $s, t \in V(S_n^q)$ for some $q \in$ <n+1> – $\{k\}$. Let $i_1 = q$ and $(v^+, u^+) \in E(S_n^k) \cap F$ such that $e_{n+1}(v^+), e_{n+1}(u^+) \notin F$. If one of v^+ or u^+ connects to $S_n^{i_1}$, then let $i_2 = k$, $u^2 = u^+$, $v^2 = v^+$. Otherwise, let $i_3 = k$, $u^3 = u^+$, $v^3 = v^+$. In the following we only consider when $k = i_3$. The case $k = i_2$ is similar to the case $k = i_3$. Without loss of generality, let (u^+, v^2), $(v^+, u^4) \in E_{n+1}(S_{n+1}) - F$, where $v^2 \in V(S_n^{i_2})$, $u^4 \in V(S_n^{i_4})$ and $i_2, i_4 \in$ <n+1> – $\{i_1, i_3\}$. Let $(v^1, u^1) \neq (s, t)$ be an edge of $S_n^{i_1}$ with (v^1, u^2), $(u^1, v^{n+1}) \in E_{n+1}(S_{n+1}) - F$, where $u^2 \in V(S_n^{i_2})$, $v^{n+1} \in V(S_n^{i_{n+1}})$, v^1 and u^+ are in different partite sets, and $i_{n+1} \in$ <n+1> – $\{i_1, i_2, i_3, i_4\}$. Since $| F \cap E(S_n^{i_1}) | = 0$, with Lemma 6, we can find an s-t path including (v^1, u^1) with a length $n! - 1$ (or $n! - 2$) when s and t are in different partite sets (or in the same partite set) in $S_n^{i_1}$. Since $| S_n^k \cap (F - \{(v^+, u^+)\})| = 2n - 7$, u^3 and v^3 are in different partite sets, with the induction hypothesis, there is an u^3-v^3 Hamiltonian path P_3 in $S_n^{i_3}$–$(F$–$\{(v^+, u^+)\})$. Let $A = \{i_4, i_5, \ldots, i_{n+1}\}$, where $\{i_5, i_6, \ldots, i_n\}$ = <n+1> – $\{i_1, i_2, i_3, i_4, i_{n+1}\}$. Since u^2 and v^2, u^4 and v^{n+1} are in different partite sets, by Lemma 1 and Lemma 5, there is an u^2-v^2 Hamiltonian path P_2 in $S_n^{i_2}$ and an u^4-v^{n+1} Hamiltonian path P in S_A–F. An s-t path in S_{n+1} is constructed as follows (see Fig. 5):

$< s, \ldots, v^1, u^2, P_2, v^2, u^3, P_3, v^3, u^4, P, v^{n+1}, u^1, \ldots, t >$.

The length of the s-t path is $(n! - 1) \times n + (n + 1) + (n! - 1) - 1 = (n + 1)! - 1$ if s and t are in different partite set, and the length of the s-t path is $(n! - 1) \times n + (n + 1) + (n! - 2) - 1 = (n + 1)! - 2$ if s and t are in the same partite set.

Case 2.5. $s \in V(S_n^q)$ and $t \in V(S_n^r)$ for some $q, r \in$ <n+1> – $\{k\}$, where $q \neq r$. Let $i_1 = q$ and $i_{n+1} = r$. Let $(v^+, u^+) \in E(S_n^k) \cap F$ such that $e_{n+1}(v^+), e_{n+1}(u^+) \notin F$. Three cases are further considered:

Case 2.5.1. Neither v^+ nor u^+ connects to $S_n^{i_1}$ or $S_n^{i_{n+1}}$. Let $i_3 = k$, $u^3 = u^+$, $v^3 = v^+$. Without loss of generality, let u^+ and s are in the same partite set. Let (v^+, u^4), $(u^+, v^2) \in E_{n+1}(S_{n+1}) - F$, where $v^2 \in V(S_n^{i_2})$, $u^4 \in V(S_n^{i_4})$, and $i_2, i_4 \in$ <n+1> – $\{i_1, i_3, i_{n+1}\}$. We can find $(v^1, u^2) \in E_{n+1}(S_{n+1}) - F$ with $v^1 \in V(S_n^{i_1})$, $u^2 \in V(S_n^{i_2})$, v^2 and u^2 are in different partite sets. Since $| F \cap E(S_n^{i_1}) | = | F \cap E(S_n^{i_2}) | = 0$, by Lemma 1, there is an s-v^1 Hamiltonian path P_1 and an u^2-v^2 Hamiltonian path P_2. Since $| S_n^k \cap (F - \{(v^+, u^+)\})| = 2n - 7$, u^3 and v^3 are in different partite sets, with the induction hypothesis, there is an u^3-v^3 Hamiltonian path P_3 in $S_n^{i_3}$–$(F$–$\{(v^+, u^+)\})$. Let $A = \{i_4, i_5, \ldots, i_{n+1}\}$, where $\{i_5, i_6, \ldots, i_n\}$ = <n+1> – $\{i_1, i_2, i_3, i_4, i_{n+1}\}$. By Lemma 5, S_A–F is strongly Hamiltonian laceable and we can find a corresponding u^4-t path P in S_A–F. An s-t path in S_{n+1} is constructed as follows (see Fig. 4):

$< s, P_1, v^1, u^2, P_2, v^2, u^3, P_3, v^3, u^4, P, t >$.

The length of the s-t path is $(n! - 1) \times n + n + (n! - 1) = (n + 1)! - 1$ if s and t are in different partite set, and the length of the s-t path is $(n! - 1) \times n + n + (n! - 2) = (n + 1)! - 2$ if s and t are in the same partite set.

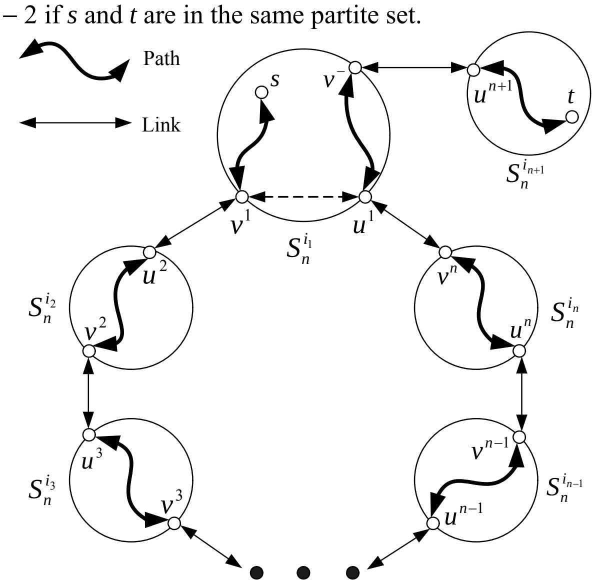

Fig.9. Illustration for case 2.5.2.

Case 2.5.2. Exactly one of v^+ or u^+ connect to $S_n^{i_1}$ or $S_n^{i_{n+1}}$. Without loss of generality, we may assume u^+ connects to $S_n^{i_1}$. Let $i_2 = k$, $u^2 = u^+$, $v^2 = v^+$, (u^+, v^1), $(v^+, u^3) \in E_{n+1}(S_{n+1}) - F$, where $v^1 \in V(S_n^{i_1})$, $u^3 \in V(S_n^{i_3})$ and $i_3 \in$ <n+1> – $\{i_1, i_2, i_{n+1}\}$ (Note that it is possible that $v^1 = s$.) Since $| S_n^k \cap (F - \{(v^+, u^+)\})| = 2n - 7$, u^2 and v^2 are in different partite sets, with the induction hypothesis, there is an u^2-v^2 Hamiltonian path P_2 in $S_n^{i_2}$–$(F$–$\{(v^+, u^+)\})$. We can find $(v^1, u^1) \in E(S_n^{i_1})$ with $(u^1, v^n) \in E_{n+1}(S_{n+1}) - F$, where $u^1 \neq s \in V(S_n^{i_1})$, $v^n \in V(S_n^{i_n})$, $i_n \in$ <n+1> – $\{i_1, i_2, i_3, i_{n+1}\}$. Let $(u^{n+1}, v^-) \in E_{n+1}(S_{n+1}) - F$ with $v^- \neq s \in V(S_n^{i_1})$, $u^{n+1} \in V(S_n^{i_{n+1}})$, v^- and t are in the same partite set. By Lemma 1 and Lemma 6, we can find an u^{n+1}-t Hamiltonian path P_{n+1} in $S_n^{i_{n+1}}$ and an s-v^- path including (v^1, u^1) with a length $n! - 1$ (or $n! - 2$) in $S_n^{i_1}$ when s and t are in different partite sets (or in the same partite set). Let $A = \{i_3, i_4, \ldots, i_n\}$, where $\{i_4, i_5, \ldots, i_{n-1}\} = \langle n+1\rangle - \{i_1, i_2, i_3, i_n, i_{n+1}\}$. Since u^3 and v^n are in different partite sets, by Lemma 5, there is an u^3-v^n Hamiltonian path P in $S_A - F$. An s-t path in S_{n+1} is constructed as follows (see Fig. 9):

$< s, \ldots, v^1, u^2, P_2, v^2, u^3, P, v^n, u^1, \ldots, v^-, u^{n+1}, P_{n+1}, t >$.

The length of the s-t path is $(n! - 1) \times n + (n + 1) + (n! - 1) - 1 = (n + 1)! - 1$ if s and t are in different partite sets, and the length of the s-t path is $(n! - 1) \times n + (n + 1) + (n! - 2) - 1 = (n + 1)! - 2$ if s and t are in the same partite set.

Case 2.5.3. Both v^+ and u^+ connect to $S_n^{i_1}$ and $S_n^{i_{n+1}}$. Without loss of generality, we may assume u^+ connects to $S_n^{i_1}$ and v^+ connects to $S_n^{i_{n+1}}$. Let $i_2 = k$. Two cases are further considered:

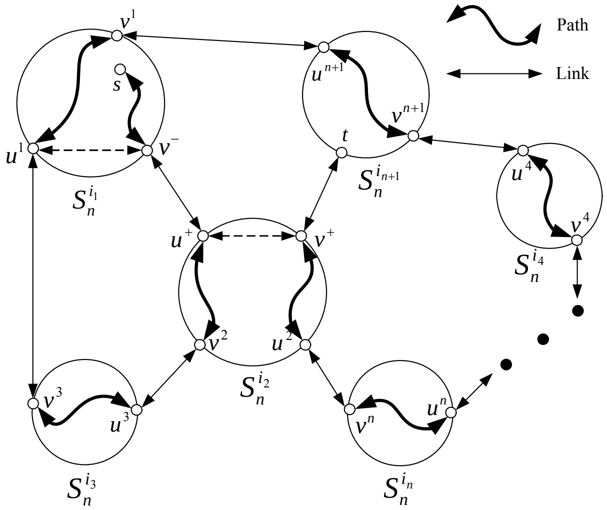

Fig.10. Illustration for case 2.5.3.1.

Case 2.5.3.1. At least one of u^+ and v^+ connects to s or t. Without loss of generality, assume v^+ connects to t. Let $(u^+, v^-) \in E_{n+1}(S_{n+1}) - F$, where $v^- \in V(S_n^{i_1})$ (It is possible that $v^- = s$). We can find $(u^1, v^-) \in E(S_n^{i_1})$ with $u^1 \neq s$, $(u^1, v^3) \in E_{n+1}(S_{n+1}) - F$, where $v^3 \in V(S_n^{i_3})$ and $i_3 \in \langle n+1\rangle - \{i_1, i_2, i_{n+1}\}$. Select v^1 and t are in the same partite set such that $(v^1, u^{n+1}) \in E_{n+1}(S_{n+1}) - F$, where $v^1 \neq s \in V(S_n^{i_1})$ and $u^{n+1} \in V(S_n^{i_{n+1}})$. Since $| F \cap E(S_n^{i_1}) | = 0$, with Lemma 6, there is a s-v^1 path including (u^1, v^-) with a length of $n! - 1$ (or $n! - 2$) in S_n if s and t are in different partite sets (or in the same partite set). Let (v^2, u^3), (u^2, v^n), $(v^{n+1}, u^4) \in E_{n+1}(S_{n+1}) - F$, where $u^m, v^m \in V(S_n^{i_m})$ for $m \in \{2, 3, 4, n, n + 1\}$, u^2 and v^2, u^3 and v^3 are in different partite sets, v^{n+1} and u^{n+1} are in the same partite set, and $i_4, i_n \in \langle n+1\rangle - \{i_1, i_2, i_3, i_{n+1}\}$ (if $n = 4$, then $i_4 = i_n$). By Lemma 1 and Lemma 2, there is a u^3-v^3 Hamiltonian path P_1 in $S_n^{i_3}$ and a u^{n+1}-v^{n+1} Hamiltonian path P_2 in $S_n^{i_{n+1}} - \{t\}$. Note that $|S_n^k \cap (F - \{(v^+, u^+)\})| = 2n - 7$, with the induction hypothesis, there is an u^2-v^2 Hamiltonian path P in $S_n^{i_2} - (F - \{(v^+, u^+)\})$. Assume that P contains (v^+, u^+) (otherwise, the discussion is easier). Let $A = \{i_4, i_5, \ldots, i_n\}$, where $\{i_5, i_6, \ldots, i_{n-1}\} = \langle n+1\rangle - \{i_1, i_2, i_3, i_4, i_n, i_{n+1}\}$. Note that v^1 and t are in the same partite set, hence v^1, u^1, u^2, u^3, u^4 are all in the same partite set (see Fig. 10), thus, u^4 and v^n are in different partite sets. By Lemma 5, we can find a u^4-v^n Hamiltonian path P_3 in $S_A - F$. An s-t path in S_{n+1} is constructed as follows:

$< s, \ldots, v^-, u^+, \ldots, v^2, u^3, P_1, v^3, u^1, \ldots, v^1, u^{n+1}, P_2, v^{n+1}, u^4, P_3, v^n, u^2, \ldots, v^+, t >$.

The length of the s-t path is $(n! - 1) \times (n - 1) - 1 + (n! - 2) + (n + 3) + (n! - 1) - 1 = (n + 1)! - 1$ if s and t are in different partite set, and the length of the s-t path is $(n! - 1) \times (n - 1) - 1 + (n! - 2) + (n + 3) + (n! - 2) - 1 = (n + 1)! - 2$ if s and t are in the same partite set.

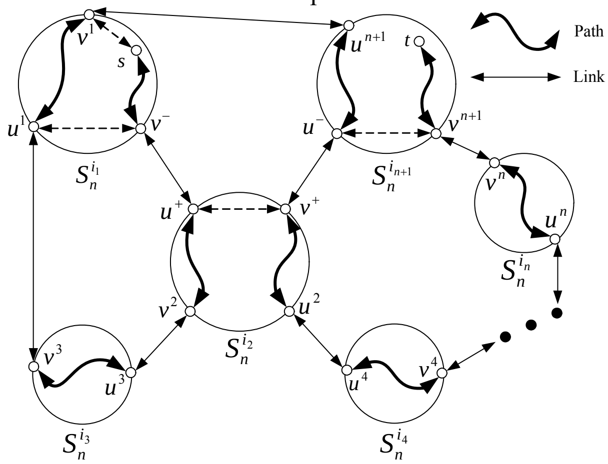

Fig.11. Illustration for case 2.5.3.2.

Case 2.5.3.2. u^+ does not connect to s and v^+ does not connect to t. Let (u^+, v^-), $(v^+, u^-) \in E_{n+1}(S_{n+1}) - F$, where $v^- \in V(S_n^{i_1})$ and $u^- \in V(S_n^{i_{n+1}})$. Since $| F \cap E(S_n^{i_1}) | = 0$, we can find (u^1, v^-), $(s, v^1) \in E(S_n^{i_1})$ such that (u^1, v^3), $(v^1, u^{n+1}) \in E_{n+1}(S_{n+1}) - F$, where $u^m, v^m \in V(S_n^{i_m})$ for $m \in \{1, 3, n + 1\}$, $u^1 \neq s$, $u^{n+1} \neq t$, and $i_3 \in \langle n+1\rangle - \{i_1, i_2, i_{n+1}\}$. Then by Lemma 7, there exists a Hamiltonian cycle including (s, v^1) and (u^1, v^-). Let (v^2, u^3), $(u^2, u^4) \in E_{n+1}(S_{n+1}) - F$, where $u^m, v^m \in V(S_n^{i_m})$ for $m \in \{2, 3, 4\}$, u^2 and v^2, u^3 and v^3 are in different partite sets, and $i_4 \in \langle n+1\rangle - \{i_1, i_2, i_3, i_{n+1}\}$. By Lemma 1, there is a u^3-v^3 Hamiltonian path P_1 in $S_n^{i_3}$. Note that $|S_n^k \cap (F - \{(v^+, u^+)\})| = 2n - 7$, with the induction hypothesis, there is an u^2-v^2 Hamiltonian path P in $S_n^{i_2} - (F - \{(v^+, u^+)\})$. Assume that P contains (v^+, u^+) (otherwise, the discussion is easier). Since $| F \cap E(S_n^{i_{n+1}}) | = 0$, we can find $(u^-, v^{n+1}) \in E(S_n^{i_{n+1}})$ such that $v^{n+1} \neq t$ and $(v^{n+1}, v^n) \in E_{n+1}(S_{n+1}) - F$, where $v^n \in V(S_n^{i_n})$ and $i_n \in \langle n+1\rangle - \{i_1, i_2, i_3, i_4, i_{n+1}\}$ (if $n = 4$, then $i_n = i_4$). Since u^{n+1} and s are in the same partite set and $| F \cap E(S_n^{i_{n+1}}) | = 0$, with Lemma 6, there is a u^{n+1}-t path including (u^-, v^{n+1}) with a length of $n! - 1$ (or $n! - 2$) in $S_n^{i_{n+1}}$ if s and t are in different partite sets (or in the same partite set). Let $A = \{i_4, i_5, \ldots, i_n\}$, where $\{i_5, i_6, \ldots, i_{n-1}\} = \langle n+1\rangle - \{i_1, i_2, i_3, i_4, i_n, i_{n+1}\}$. Note that u^+, u^-, u^1, u^3, u^2, v^n are all in the same partite set (see Fig. 11). Thus, u^4 and v^n are in different partite sets. By Lemma 5, we can find a u^4-v^n Hamiltonian path P_2 in $S_A - F$. An s-t path in S_{n+1} is constructed as follows:

$< s, \dots, v^-, u^+, \dots, v^2, u^3, P_1, v^3, u^1, \dots, v^1, u^{n+1}, \dots, u^-, v^+, \dots, u^2, u^4, P_2, v^n, v^{n+1}, \dots, t >$.

The length of the s-t path is $2 \times ((n! - 1) - 1) + (n! - 1) \times (n - 2) + (n + 3) + ((n! - 1) - 1) = (n + 1)! - 1$ if s and t are in different partite set, and the length of the s-t path is $2 \times ((n! - 1) - 1) + (n! - 1) \times (n - 2) + (n + 3) + ((n! - 2) - 1) = (n + 1)! - 2$ if s and t are in the same partite set. □

The result above is optimal. It means for any S_n ($n \geq 4$) with $\geq 2n - 6$ edge faults in which each node is incident to at least two healthy edges, we probably could not find a fault-free Hamiltonian path between two arbitrary nodes in different partite sets. The proof is in the following theorems.

Theorem 2. Let $F \subset E(S_n)$, $|F| \geq 2n - 6$ and $\delta(S_n - F) = 2$, where $n \geq 4$. Then $S_n - F$ might not be Hamiltonian laceable.

Proof. When $|F| = 2n - 6$, consider the condition in Fig. 12, where dashed edges denote edges in F and real ones denote edges not in F. In S_n, each node has degree $n - 1$. Suppose (u, x), (x, y), $(y, v) \notin F$ and the other edges incident to x or y are in F, so $|e(x) \cap F| = n - 3 = |e(y) \cap F|$. Now let u be the starting point and v is the ending point. It is clear that u and v are in different partite sets. Since $< u, x, y, v >$ is the only path from u to v in $S_n - F$ pass though x or y, it can't be Hamiltonian laceable. □

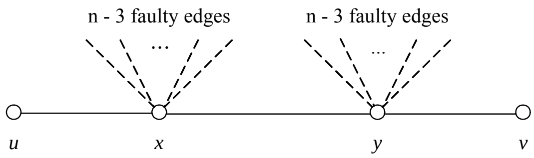

Fig.12. Illustration for Theorem 2.

4: CONCLUDING REMARKS

Since processor or link faults may occur in real world networks, fault tolerance is an important research subject of the multi-process computer systems. We have shown that for any S_n ($n \geq 4$) with $\leq 2n - 7$ edge faults in which each node is incident to at least two healthy edges, a fault-free Hamiltonian path could be obtained between two arbitrary nodes in different partite sets, and a fault-free path with length $n! - 2$ could be obtained between two arbitrary nodes of the same partite set. Our result demonstrates that S_n is powerful in fault-tolerant Hamiltonicity. Since there exists an S_n ($n \geq 4$) with $2n - 6$ faulty edges, in which each node is incident to at least two healthy edges, and for which there is no Hamiltonian path between some node in different partite sets, this result is optimal.

Acknowledgment

The authors would like to thank the National Science Council of the Republic of China, Taiwan for financially supporting this research under Contract No. NSC 95-2221-E-239-002-.

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