Strong Menger Connectivity on the Class of Hypercube-like Networks.

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Abstract

Motivated by some research works in networks with faults, we are interested in the connectivity issue of a network. Suppose that a network G has a set F of faulty vertices and let G−F is the resulting network with those vertices in F removed. We say that a k-regular graph G is strongly Menger connected if each pair u and v of G−F are connected by $min\{deg_f(u), deg_f(v)\}$ *vertex-disjoint fault-free paths in G−F, where* $deg_f(u)$ *and* $deg_f(v)$ *are the degree of u and v in G − F, respectively. In this paper, we consider an n-dimensional Hypercube-like networks* HL_n. *We show that for each pair u and v in* HL_n-F, *where F is a set of vertices, for* $|F| \leqq n-2$, *there are* $min\{deg_f(u),deg_f(v)\}$ *vertex-disjoint fault- free paths connecting u to v, for* $n \geqq 3$.

1: Introduction

For the graph definitions and notations we follow [8]. A graph is denoted by G with the vertex set $V(G)$ and the edge set $E(G)$. A graph G is *connected* if there is a path between any two distinct nodes. A subset S of $V(G)$ is a *cut set* if $G - S$ is disconnected. The *connectivity* of G, written $\kappa(G)$, is defined as the minimum size of a vertex cut if G is not a complete graph, and $\kappa(G) = |V(G)| - 1$ if otherwise. A graph G is *k-connected* if $k \leqq \kappa(G)$. We say that a graph has *connectivity* k if it is k-connected but not $(k + 1)$-connected. In this paper, we study the Hypercube-like networks, and show that it has a stronger connectivity property.

Let $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ be two disjoint graphs with the same number of vertices. A *one-to-one connection* between G_0 and G_1 is defined as an edge set $M = \{(v, \psi(v)) \mid v \in V_0, \psi(v) \in V_1 \text{ and } \psi: V_0 \rightarrow V_1 \text{ is a bijection}\}$. We use $G_0 \oplus_M G_1$ to denote $G=(V_0 \cup V_1, E_0 \cup E_1 \cup M)$. The operation "$\oplus_M$" may generate different graphs depending on the bijection ψ. Chen et al. [2, 3] showed that the connectivity of $G_1 \oplus_M G_2$ is increased to $k+1$, where both G_1 and G_2 have connectivity k. By the classical Menger's Theorem [7], if a network G is k-connected, then every pair of vertices in G are connected by k vertex-disjoint paths. We can infer that there are $k+1$ vertex-disjoint paths between u and v, where u and v are vertices in $G_1 \oplus_M G_2$.

The degree of a vertex u, denoted by $deg(u)$, is the number of vertices adjacent to u. Suppose that the network G has a set F of faulty vertices and let $G - F$ be the resulting network with those vertices in F removed. Let u and v be two fault-free vertices in $G - F$. The number of fault-free neighbor of u and v is denoted by $deg_f(u)$ and $deg_f(v)$, respectively. These values restrict the maximum number of vertex- disjoint fault-free path between u and v. It is clearly that the number of vertex-disjoint fault-free path between u and v is smaller than $min\{deg_f(u), deg_f(v)\}$. OH et al. [9] gave a definition to extend the Menger's Theorem as following:

Definition 1. [9] *A k-regular graph G is strongly Menger-connected if for any copy G − F of G with at most k − 2 vertices removed, each pair u and v of G − F are connected by* $min\{deg_f(u), deg_f(v)\}$ *vertex-disjoint fault-free paths in G − F, where* $deg_f(u)$ *and* $deg_f(v)$ *are the degree of u and v in G − F, respectively.*

This property is called *strongly Menger connected* property. OH et al. [9, 10] showed that the Star graph S_n with at most $n - 3$ vertices removed still possess the strongly Menger connected property. In this paper, we are interested in the strict bound on the size of the faulty vertex set F such that there are $min\{deg_f(u), deg_f(v)\}$ vertex-disjoint fault-free paths connecting u to v, for each pair u and v in $G - F$.

There are many useful topologies proposed in interconnection networks. Among them, the hypercube network is one of the popular ones. Various networks are proposed by twisting some pairs of links in hypercubes [1, 6, 4, 5]. To make a unified study of these variants, Vaidya et al. [14] offered a class of graphs, called a class of hypercube-like graphs. The class of Hypercube-like networks consists of simple, connected and undirected graphs, and contains most of the hypercube variants. Park et al. [12, 13] showed some properties of Hypercube-Like Networks.

We now give a recursive definition of the n-dimensional Hypercube-like networks HL_n as follows:

(1)$HL_0 = \{k_1\}$, where k_1 is a trivial graph in the sense that it has only one vertex.

(2)$G \in HL_n$ if and only if $G = G_0 \oplus_M G_1$ for some $G_0, G_1 \in HL_{n-1}$

By the above definitions if G is a graph in HL_n, then $G = G_0 \oplus_M G_1$ with both G_0 and G_1 in HL_{n-1}, for $n \geqq 1$. Let u be a vertex in $V(G_0)$. The vertex u has only one neighbor in $V(G_1)$. The connectivity of an n-dimensional Hypercube-like networks HL_n is n. In this paper, we shall show that there are $min\{deg_f(u), deg_f(v)\}$ vertex-disjoint fault-free paths connecting u to v, for each pair vertices u and v in $HL_n - F$, where F is a set of vertices with $|F| \leqq n-2$. This result is optimal in the sense that the result can not be guaranteed, if there are $n-1$ faulty vertices. For example, take an edge (x, y) and a vertex z different from x and y in HL_n. Suppose that all the $(n-1)$ vertices adjacent to x are faulty. Then $deg_f(y) = deg_f(z) = n$, (See Fig. 1). However, the number of vertex-disjoint paths between y and z is at most $n-1$, and hence there do not exist n vertex-disjoint paths connecting y and z.

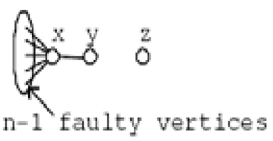

Figure 1: four different output states.

2: Some Preliminaries

To prove our main theorem, we need the following fact.

Lemma 1. *Let HLn be an n-dimensional Hypercube-like networks with* $n \geqq 3$, *and T be a set of vertices of* HL_n *such that* $|T| \leqq 2n-3$. *Then* $HL_n - T$ *satisfies either (1)* $HL_n - T$ *is connected or (2)* $HL_n - T$ *has two components, one of which is a trivial graph.*

Proof. We prove this statement by induction on n. We check this result for $n=3$ by brute force. Assume the lemma holds for $n-1$, for some $n \geqq 4$, we shall show that it is true for n. As we mentioned before, we assume that $G = G_0 \oplus_M G_1$ in HL_n. So $G_i \in HL_{n-1}$ for $i = 0, 1$. The connectivity of an n-dimensional Hypercube-like networks HL_n is n, and HL_{n-1} has connectivity $n-1$. Thus, both G_0 and G_1 are $(n-1)$-connected. Let T_0 and T_1 be a set of faulty vertices of G_0 and G_1, respectively. By assumption, $|T_0| + |T_1| = |T| \leqq 2n-3$. The proof is divided into three major cases:

Case **1**: $|T_0| \leqq n-2$ and $|T_1| \leqq n-2$. Since G_0 and G_1 are both $(n-1)$-connected, then $G_0 - T_0$ and $G_1 - T_1$ are connected. There are 2^{n-1} edges between G_0 and G_1. For $n \geqq 4$, since $2^{n-1} - 2(n-2) \geqq 1$, there is at least one edge with both ends fault-free remaining between $G_0 - T_0$ and $G_1 - T_1$. Hence $HL_n - T$ is also connected.

Case **2**: $n-1 \leqq |T_0| \leqq 2n-5$ or $n-1 \leqq |T_1| \leqq 2n-5$. Without loss of generality, we assume $n-1 \leqq |T_0| \leqq 2n-5$, then $|T_1| \leqq n-2$. So $G_1 - T_1$ is connected. We then consider that $G_0 - T_0$ is either connected or has two components, one of which has exactly one vertex. Assume first that $G_0 - T_0$ is connected. Then by the same reason of *Case 1*, $G_0 - T_0$ is connected to $G_1 - T_1$, since $2^{n-1} - (2n-3) \geqq 1$ with $n \geqq 4$. Thus, $HL_n - T$ is also connected. On the other hand, if $G_0 - T_0$ is not connected, by the induction hypothesis, $G_0 - T_0$ has two components, C_1 and C_2, with C_1 having only one vertex, we denote the vertex by x. For $x \geqq 4$, since $2^{n-1} - 1 - (2n-3) \geqq 1$, it means that there is at least one edge between C_2 and $G_1 - T_1$. Hence C_2 is connected to $G_1 - T_1$. If x (component C_1) is connected to $G_1 - T_1$, then $HL_n - T$ is also connected. Otherwise, x is not connected to $G_1 - T_1$, then $HL_n - T$ is disconnected. We conclude that it has two components, one of which is trivial graph.

Case **3**: $|T_0| \geqq 2n-4$ or $|T_1| \geqq 2n-4$. Without loss of generality, we assume $|T_0| \geqq 2n-4$. Since $|T| \leqq 2n-3$, we have either $|T_0| = 2n-4$ or $|T_0| = 2n-3$. If $|T_0| = 2n-3$, since every vertex of G_0 has a neighbor in G_1, then $HL_n - T$ is connected. Otherwise, we consider the last case $|T_0| = 2n-4$. Since $|T| = 2n-3$ and $|T_0| + |T_1| = |T|$, so $|T_1| = 1$ and there is only one faulty vertex in G_1, denoted by s. Let C be a connected component in $G_0 - T_0$. If C has at least two vertices, then C has at least two neighbors in G_1. Since s is the only one vertex in G_1, it infers that C is connected to $G_1 - T_1$. If C has only one vertex, denoted by t, then C is connected to $G_1 - T_1$ unless s is a neighbor of t. By the definition of Hypercube-like networks, s has at most one neighbor in G_0, so $HL_n - T$ is connected or $HL_n - T$ has exactly two components, one of which has exactly one vertex. □

3 Main Theorem

Our main results are presented in this section. Before proving the main theorem, we state the Menger Theorem.

Theorem 1. [7] *If x,y are vertices of a graph G and $(x, y) \notin E(G)$, then the minimum size of an x,y-cut equals the maximum number of pairwise internally disjoint x,y-paths.*

We now show that a Hypercube-like networks has a stronger connectivity property, it is strongly Menger-connected.

Theorem 2. *Consider an n-dimensional Hypercube-like networks HL_n, for $n \geqq 3$. Let F be a set of faulty vertices with $|F| \leqq n-2$. Then each pair vertices u and v of $HL_n - F$ are connected by $\min\{deg_f(u), deg_f(v)\}$ vertex-disjoint fault-free paths, where $deg_f(u)$ and $deg_f(v)$ are the degree of u and v in $HL_n - F$, respectively.*

Proof. We can assume without loss of generality that $deg_f(u) \leqq deg_f(v)$, so $\min\{deg_f(u), deg_f(v)\} = deg_f(u)$. To prove that each pair vertices u and v of $HL_n - F$ are connected by $deg_f(u)$ vertex-disjoint fault-free paths, we show that u is connected to v if the number of vertices deleted is smaller than $deg_f(u) - 1$ in $HL_n - F$.
Suppose on the contrary that u and v is separated by deleting a set of vertices V_f, where $|V_f| \leqq deg_f(u)-1$. Obviously, $|deg_f(u)-1| \leqq |deg(u)-1| \leqq n-1$. So $|V_f| \leqq n-1$. We sum the cardinality of these two sets F and V_f. Since $|F| \leqq n-2$ and $|V_f| \leqq n-1$, then $|F| + |V_f| = |T| \leqq 2n-3$. By Lemma 1, $HL_n - T$ is either connected or has two components, one of which is a trivial graph, for $|T| \leqq 2n-3$. If $HL_n - T$ has two component and one of which has only one vertex, the set V_f has to be the neighbor of u and $|V_f| = deg_f(u)$, which is a contradiction. Thus, u is connected to v when the number of vertices deleted is smaller than $deg_f(u)-1$ in $HL_n - F$.
This completes the proof. □

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