Using Computer-Assisted Instruction for the Visualization of Proof Tree to Improve the Reading Comprehension of Geometry Proofs

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ABSTRACT

Geometry has long been a nightmare for both instructors and students. This research focuses on how to help students study geometry from the perspective of reading and understanding. For most students, to comprehend the formal proof in textbooks is indeed difficult. As a result, this research introduces a computer-assisted learning environment, which can automatically draw the figure for a geometry problem, provide an interactive proof tree and a formal proof. Users of this learning environment can better understand the invariant properties of a geometric figure by dragging its points, or click on any node on the proof tree to observe the conditions supporting the node. The main purpose of this research is to facilitate students' understanding in geometric properties so that they can produce correct geometry proof later on.

1: Introductions

The assessment was performed in 2006 with 1,662 individuals taking the test. The test included questions on basic geometry, logical reasoning, numbers and quantity, basic algebra, probability and statistics. The results of the assessment show that the average score of the participants is 9.7 (within a reasonable scope), which is close to that of the students in the U.S. It is found that students in Taiwan are better at calculating than the students in the U.S. while are poorer at plane or solid geometry and reasoning (http://www. epochtimes.com/b5/6/3/16/n1256557.htm). According to Cheng [5], even though students in Taiwan know that formal deduction is important in examination, only one-third of them are able to write geometry proofs correctly. The phenomena of other countries are also not good [29,30,13]. Geometry theorem proving is one of the most challenging skills for students to learn in secondary school mathematics [4,13]. Duval [9] points out that geometry instruction is often more complex than those on arithmetic and algebra. Obviously, there is still room for students to improve their performance on geometry and reasoning, which motivates us to develop a computer-assisted environment for learning geometry.

Comparing *Curriculum and Evaluation Standards for School Mathematics* [24] and *Principles and Standards for School Mathematics* [25] issued by NCTM (National Council of Teachers of Mathematics), Knuth [17] found that the latter put more emphasis on formal deduction. This article focuses on theorem proving problems.

Before students feel comfortable in constructing proofs, they need to read and understand many examples of proofs. In traditional instruction, teachers explain some geometry proofs from textbooks in class, and students need to read many other proofs in textbooks by themselves. In a textbook, a geometry question, its geometric figures, and its proof are often presented side by side. However, a textbook does not explain each proof step as clearly as a teacher does. For example, when explaining a geometry theorem and its formal proof, a teacher can repeat the explanation or give more details, depending on how students react. The textbook is obviously weak in this aspect. Therefore, students are often faced with difficulties when they study geometry proofs by themselves. They need help.

Students encounter several common difficulties when learning geometry proofs. First of all, the studies by Moore [23] and Laborde [19] found that some students could not understand or did not know how to use mathematical language or symbols. The geometry that covers all involved theorems and definitions in the proof is often too complicated and over whelming for students to understand. Some students can not deal with too much information at the same time.

Second, students are often perplexed by the question: why a geometry proof starts the way it does. When reading a proof, a student often has difficulty differentiating the given premises and the goal proposition, which is to be proved. When students cannot understand the rational of the proof, they often end up memorizing the proof.

Third, students rely too much on typical instances of geometric concepts. In order to facilitate teaching, teachers usually demonstrate geometric concepts with typical examples. A concrete or typical geometric figure might help a novice understand geometric concepts and the key ideas in a geometry proof, but it can also hinder more advanced geometric thinking.

Fourth, student might fail to relate each proof step in a textbook proof to the accompanying figure. In textbook, a geometry proof is often presented with a single figure. When they read the proof line by line, they might fail to identify which part of the figure each proof step refers to.

Fifth, students often rely on superficial visual features, such as the guess that two angles look the same in a diagram, even if they cannot be shown to be so by reasoning logically from theorems and definitions. Such heuristics are often successful, but are likely to fail in more complex problems [1]. In a geometry proof, students will regard measurement and their guesses as justifications for establishing a condition. When a figure is not drawn precisely, students will be misled by the incorrect figure and make incorrect inferences. Students encounter several common difficulties when learning geometry proofs.

Regarding the above difficulties students confronted in learning geometry proof, a variety of proposals have been suggested by researchers. Some researchers have developed interactive learning environments for learning geometry, with results that are widely recognized. Examples include Geometer's Sketchpad [16], Cabri Geometry [20], Geometry Expert [10], Cinderella [28], Géométrix [11], Geometry Explorer [15]. These systems do bring new methods for teaching and learning geometry theorem proving when compared to traditional geometry education [8,12].

Logically speaking, a geometry proof is a tree structure. In order to help students in reading proofs, we emphasize the visualization of the steps of a proof and the strategy of teaching students to understand a proof as a tree structure. Take math problem solving for example, many researchers have placed an increased emphasis on the use of visual reasoning in mathematics [2,9,21,31]. Diagrammatic teaching plays a key role in a novice's learning of geometry proofs and helps the learner understand the meanings of theorems and diagrams.

The main purpose of this paper is to propose a computer-assisted environment to help a learner understand proof. This should help the learner to understand the proof as a tree structure rather than a linear structure presented in textbook. In the following section, we first point out the steps of understanding a proof. The third section explains how an interactive proof tree and the textbook help students understand the theorem proving process. The fourth section discusses the architecture and functions of the system. The conclusion of this research and future work are presented in the last section.

2: Reading Comprehension of Geometry Proofs

We use the Yang [32] to explain the cognitive process of reading a geometry proof. The process involves superficial comprehension, referring, association, and transformation. First of all, a reader uses her prior knowledge to understand the mathematical language in the proof (superficial comprehension). She also needs to differentiate the given conditions from the goal to be proved by referring to the given figure or create her own figure (referring). The reader can also relate the mathematical language of the problem to relevant knowledge (association). If she can recognize a theorem used in the proof by identifying the premises and conclusion of the theorem (transformation), she has a better chance of understanding the problem fully. In short, reading and understanding a geometry proof is an activity that requires various skills in multiple steps. However, most textbooks do not satisfy learners' needs in their reading process.

Effective reading is important for studying geometry. This research aims to focus on a specific issue of geometry theorem proving—reading and understanding geometry proofs. Many studies on reading comprehension focus on language learning but few deals with reading proofs. Yang [32] pointed out that there had not been complete and in-depth studies on reading and understanding geometry proofs, which remains an open research topic.

3: Proof Organized As a Tree Structure

3.1 Solution Tree to Math Problem Solving

For a long time researchers have been studying computer-assisted math learning environments which can provide diagram to help students solve math problems. Reusser [27] proposed a system called HERON that uses schema representation to develop plans and produce a solution tree. Students, from grades 3 through 9, can understand many mathematical problems through various schemata, and then describe the solution steps in detail by constructing a solution tree. It can detect mistakes in a student's solution tree. A similar system MathCAL [3], sought to increase the detection accuracy of students' mistakes. In the stage of "plan making", which is one of Polya's four problem-solving stages [26], the representation of schemata and solution tree can help students solve a problem systematically by following the steps of a plan. The system was empirically demonstrated to be effective in improving the performance of low-achievement students. In short, the solution tree can help students understand and solve complicated mathematical problems.

ANGLE [18] includes a graphical editor for entering geometry "flow proofs", a cognitive model or "expert component" for checking student proofs, and a tutor for providing feedback and advice on student's work. A proof is represented as a graph that shows how the problem goal is supported by a chain of inferences from the given conditions. In Advanced Geometry Tutor (AGT) [22], a solution graph is also used to represent a proof. AGT employs an efficient and semi-complete procedure for constructing proofs. Either of two problem-solving strategies can be used, namely forward chaining and backward chaining. Both ANGLE and AGT focus on learner's construction of proofs, while our system focuses on learner's understanding of proofs constructed by the system.

3.2 Representing a Geometry Proof as a Tree

A proof tree (Figure 1(c)) is the result of the theorem proving process. The root node of the tree is the goal condition to be proved. The tree consists of *leaf nodes* (for example, two segments of the same length, two angles of the same measure) and *derived nodes*. A new node is derived from children nodes based on some theorem (for example, two triangles are congruent due to the facts that their corresponding sides are equal in length, according to the SSS theorem). Each node of a proof tree represents a proof step. Each leaf node is a given fact or self-evident condition, e.g., a segment's length equals that of itself.

A proof tree (Figure 1(c)) is the logical structure underlying a common textbook proof (Figure 1(a) and 1(b)). In the proof tree, the logical relation between the given facts and inferred propositions are made explicit and easier to understand. This structure helps students understand the flexibility of ordering the proof steps in a common textbook proof. In a purely bottom-up presentation of the proof tree, the proof tree shows clearly that a proof must be built on the given conditions of the problem. The ultimate goal is to infer the proposition of the root node. All other inferred propositions are needed in order to lead to the root conclusion. In the proof tree, the children nodes provide sufficient conditions for establishing their parent node. By studying the proof tree, students can observe and understand the global structure of a proof.

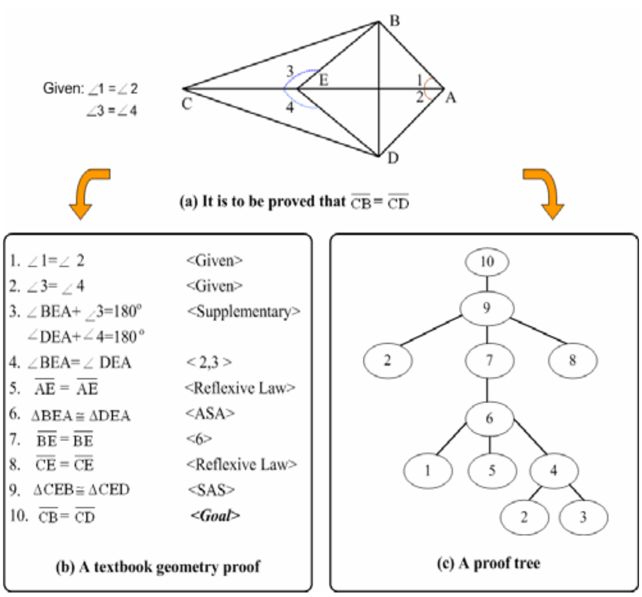

Fig. 1 Linear proof and Proof tree

Proof tree shows all steps of how to derive the goal. Figure 1(b) is a linear sequence, which is a typical textbook proof in middle school, of presenting the proof tree in Figure 1(c). In fact, a proof tree may be presented linearly with different orderings of the nodes. A proof tree can be considered as a partial ordering relation between the nodes. A linear presentation of the tree's nodes can be considered a scheduling task. In Figure 1(b) *Proposition 2* has a higher priority than *Proposition 4*, which means that *Proposition 2* must be presented prior to *Proposition 4*. The *Proposition 1, Proposition 5* and *Proposition 4* support *Proposition 6*, which means that *Proposition 1, Proposition 5* and *Proposition 4* should be presented before *Proposition 6*. A proof tree expresses clearly the logical relation between each parent node and its children nodes. If the student studies these relations carefully, she should be able to understand the proof fully.

4: Overview of the System Architecture and Design

4.1 System Architecture

We have developed a system that does automated theorem proving and visualize the steps of a proof, aiming to facilitate students' reading and comprehension of geometry proofs. In this section, the architecture and the design principles of our system are discussed. The system components includes an inference engine, knowledge base of inference rules, and a Prolog geometry prover, a geometry visualizer driven by Geometry Script Language (GSL) which describes the conditions of a geometric figure, GeoProver [14], and proof tree visualizer. The system provides the basic features for visualizing dynamic geometry and proving theorems mechanically with rule-based and algebraic approaches. Algebraic computations are done with Maple, which talks with a Java application. The system architecture is shown in Figure 2.

Maple is a popular, commercial computer algebra system (http://www.maplesoft.com/). Maple can obtain exact analytical solutions to many mathematical problems, including integrals, systems of multivariate equations, differential equations, and problems in linear algebra.

Prolog is a programming language based on a small set of mechanisms, including pattern matching, tree-based data structure and automatic backtracking. We can write short Prolog programs to reason about the spatial relationships between geometric objects and check their consistency with respect to general rules. These features make Prolog a powerful language for artificial intelligence and non-numerical programming in general.

We use JLog to be the Prolog kernel of our system. JLog (http://jlogic.sourceforge.net), written in pure Java, is developed by Glendon Holst as an open source project. Its primary advantage is that it can be run on almost any platform supporting Java (with or without a web browser), and as such it is well suited for educational purposes. We reconstruct a new user interface for our purpose and build more predicates with JLog. Our system does algebraic operations in Maple, then returns the result to Java, and finally shows the result in a Prolog session.

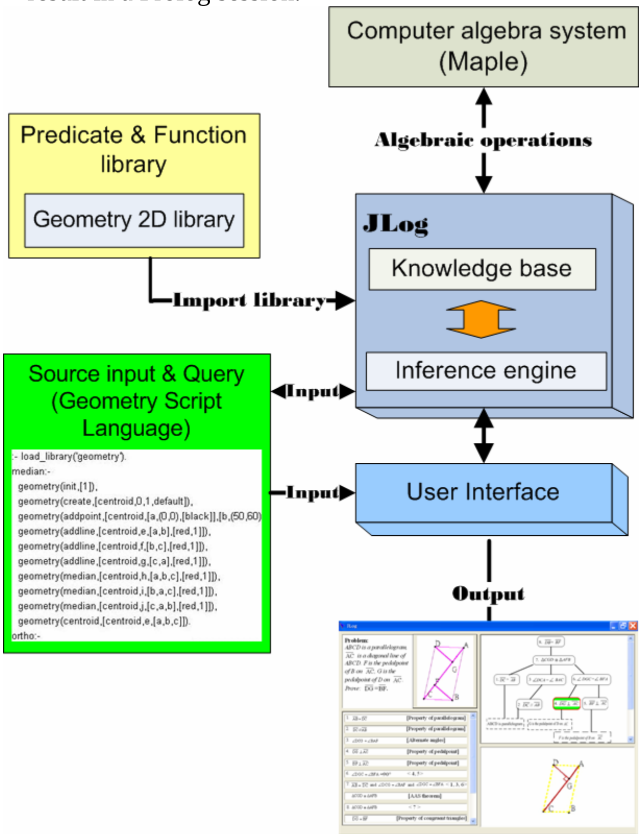

Fig. 2 System architecture

The GeoProver package is originally designed to prove geometry theorems with an algebraic approach. The system employs GeoProver to compute the coordinates of all points derived from "free" points or other derived points. It contains many generic proof schemes of geometry theorems, mainly from Chou's book and dissertation [6,7]. GeoProver can translate all geometric statements into algebraic formulas and try to solve the corresponding algebraic problem by algebraic methods.

Instead of using GeoProver to prove theorems, our system uses a rule-based method, which is more suitable for middle and high school students. The rules can be divided into two categories—geometric definition (for example, the midpoint of a line segment bisects the segment) and geometric theorems (for example, the triangle congruence theorems SSS, AAS, etc.). The system can automatically infer the goal proposition based on the rules. After inferencing, the system produces a proof tree. The root node is the goal to be proved and the leaves are the given facts, or self-evident properties. Other nodes are the propositions derived in the inferencing process. The system employs GeoProver to compute the algebraic coordinates of all points for visualization.

4.2 Visualization of Proof Tree

With a rule-based approach, propositions can be inferred in two directions: data-driven forward chaining and goal-driven backward chaining. The system uses logic programming to implement backward chaining, with depth-first search and backtracking. After receiving a goal proposition, the system carries out a goal-driven search in the knowledge base. If the goal proposition matches a given condition then the goal is proved. Otherwise, if a rule whose condition matches the goal is found, the system will try to prove each premises of the rule recursively.

If the goal is proved, all proof steps in the inferencing process will be collected and returned as a proof tree. The learner can choose any node in the proof tree to visualize while hiding other parts in the figure. The system will draw the geometric figure with solid lines representing the condition of the parent node and dash lines representing those of children nodes in the figure. The lines are colored to distinguish the parent and children nodes.

4.3 An Example

Problem: *ABCD is a parallelogram,* $\overline{AC}$ *is a diagonal line within ABCD. F is the pedalpoint of B on* $\overline{AC}$ *, G is the pedalpoint of D on (Figure 3). Prove:* $\overline{DG}=\overline{BF}$.

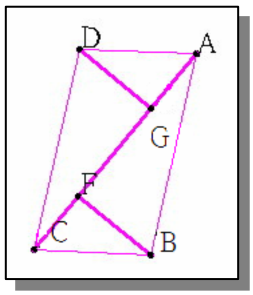

Fig. 3 The graph of the example

The steps of the proof can be represented as a tree structure (Figure 4). From this tree structure, the facts needed for proving the goal are clearly shown as the leaf nodes. The next section explains how a proof tree can be derived mechanically.

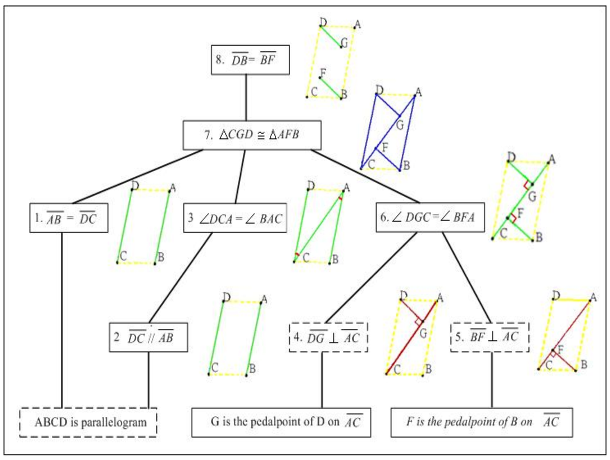

Fig. 4 Proof tree

4.4 GUI components

Problem Description: Figure 5 shows a geometry problem, where the system will draw the corresponding geometric figure. One feature of this system is that a student can explore various sceneries including atypical ones, by dragging some "free" points of the figure. She can observe different consequences under the same given conditions. In this way, students can be better trained in recognizing geometric figures. Figure 6 shows that the pedalpoint from a vertex of the parallelogram onto the diagonal can lie inside or outside the parallelogram.

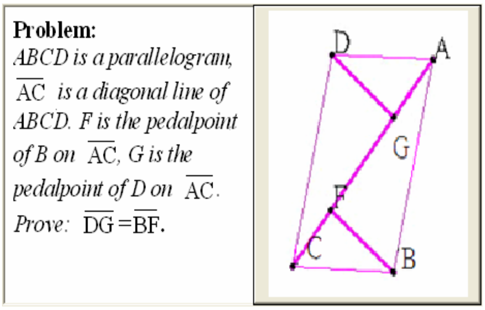

Fig. 5 Problem Description

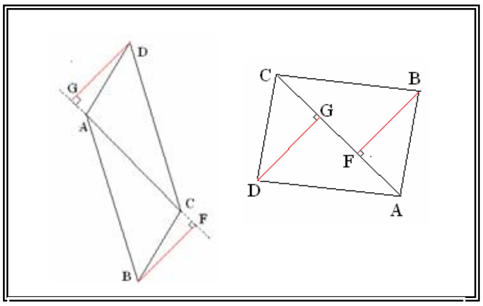

Fig. 6 The invariant properties of geometric figures

Linear Proof: The Linear Proof area in Figure 7 shows a complete proof, which is mechanically generated by the inference engine. A proof is realized with a two-column format. Each row specifies the sufficient conditions for deriving a proposition with a theorem. Each sufficient condition can be a given fact or a proposition derived earlier.

1. $\overline{AB} = \overline{DC}$	[Property of parallelogram]
2. $\overline{DC} // \overline{AB}$	[Property of parallelogram]
3. $\angle DCG = \angle BAF$	[Alternate angles]
4. $\overline{DG} \perp \overline{AC}$	[Property of pedalpoint]
5. $\overline{BF} \perp \overline{AC}$	[Property of pedalpoint]
6. $\angle DGC = \angle BFA = 90°$	< 4, 5 >
7. $\overline{AB} = \overline{DC}$ and $\angle DCG = \angle BAF$ and $\angle DGC = \angle BFA$	< 1, 3, 6 >
$\Delta CGD \cong \Delta AFB$	[AAS theorem]
8. $\Delta CGD \cong \Delta AFB$	< 7 >
$\overline{DG} = \overline{BF}$	[Property of congruent triangles]

Fig. 7 Linear Proof

Proof Tree: With a click on any node on the proof tree, a student can see the corresponding proof step highlighted. This will help students in their understanding of the correspondence between the linear proof and the proof tree. The proof tree clearly shows the relationship between each parent and its children nodes (see Figure 8).

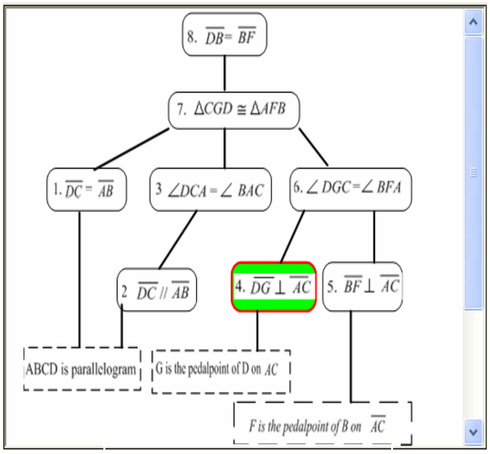

Fig. 8 Proof Tree

Highlight Features: When the student clicks at a node on the tree, the highlight area will draw the corresponding features of the geometric figure with solid lines representing the condition of the parent node and dash lines representing those of the children nodes in the figure. While the student reads the proof tree or the linear proof, she can also refer to the highlight feature of the figure.

5: Conclusion

Many learners find it difficult to understand all steps of a proof when reading the proof, since many skills are involved. We address this issue by designing a computer-assisted environment which uses a theorem proving engine to produce proof trees and visualizes them. The system consists of an inference engine, a knowledge base of inference rules, and the geometry prover GeoProver and a dynamic geometry tool. The learner can choose any node of a proof tree to highlight the geometric features involved in the corresponding proof step. By focusing on one proof step at a time, the student would have a better chance in understanding what sufficient conditions and which theorem are used to establish the proposition in this step.

In the future, we will design geometry materials to evaluate how students perform in their reading comprehension of geometry proofs after using the system. Evaluation results will be used to improve the system further. When the system is used, all exploratory actions of a user can be recorded and assistance will be provided to guide the learner to understand a proof in depth.

6: Acknowledgements

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